Published by Institute of Physics Publishing for SISSA

Received: *March 5, 2005*
Revised: *August 4, 2005*
Accepted: *September 15, 2005*
Published: *October 10, 2005*

Matching parton showers to NLO computations

Zoltán Nagy

Institute for Theoretical Physics, University of Zürich
Winterthurerstrasse 190, CH-8057 Zürich, Switzerland
E-mail: nagyz@physik.unizh.ch

Davison E. Soper

Institute of Theoretical Science, University of Oregon
Eugene, OR 97403-5203, U.S.A.
E-mail: soper@physics.uoregon.edu

Abstract: We give a prescription for attaching parton showers to next-to-leading order (NLO) partonic jet cross sections in electron-positron annihilation. Our method effectively extends to NLO the scheme of Catani, Krauss, Kuhn, and Webber for matching between m hard jets and $(m+1)$ hard jets. The matching between parton splitting as part of a shower and parton splitting as part of NLO matrix elements is based on the Catani-Seymour dipole subtraction method that is commonly used for removing the singularities from the NLO matrix elements.

Keywords: Jets, QCD, NLO Computations.

JHEP10(2005)024

Contents

JHEP10(2005)024

1. Introduction

One often uses perturbation theory to produce predictions for the results of particle physics experiments in which the strong interaction is involved. Let us suppose that the measurement to be made is cast in the form of the cross section weighted by a function $F(f)$ that assigns a number to each possible final state f. The object then is to predict $\sigma[F]$. Perturbation theory can be relevant if a high momentum transfer Q is involved in the reaction. In that case, one arranges the calculation so that one is performing an expansion in powers of $\alpha_s(Q)$, which is small when Q is large even though $\alpha_s(1\,\mathrm{GeV})$ is not small. If there are hadrons in the initial state, then low scale effects related to the initial state must be factored into parton distribution functions. Low scale effects related to the final state can be avoided if F is "infrared safe," as described in section 2 and if its definition does not involve any small parameters.

It will prove helpful to have an example, taken from $e^+e^- \to hadrons$. Let $F(f)$ be $(1-t_f)^4$ where t_f is the thrust of the particles in f. If we call this observable T_4, we have

$$\sigma[T_4] = \int_0^1 dt\,(1-t)^4\,\frac{d\sigma}{dt}\,. \tag{1.1}$$

A related example is obtained by collecting the particles in f into three jets using a suitable jet algorithm and defining m_f to be the mass of the most massive of the three jets. Then we can take $F(f)$ to be $dT_4/dM^2 = (1-t_f)^4\delta(m_f^2 - M^2)$, so that

$$\sigma\left[\frac{dT_4}{dM^2}\right] \equiv \frac{d\sigma[T_4]}{dM^2} = \int_0^1 dt\,(1-t)^4\,\frac{d\sigma}{dt\,dM^2}\,. \tag{1.2}$$

For fixed M^2/s, this is an infrared safe observable, but if $M^2/s \ll 1$, its perturbative expansion contains large logarithms, $\log(M^2/s)$. Quantities like this need a treatment beyond a fixed order of perturbation theory. In practical applications one often needs not only a treatment of the large logarithms but also a good model for how partons turn into hadrons. That is, to make predictions for $\sigma[T_4]$, we need just short distance physics, but to understand $d\sigma[T_4]/dM^2$ for all M^2, we need to model also long distance physics.

The most widely used tools available for perturbative calculations can be classified into two groups. The first is parton shower Monte Carlo (MC) event generators, such as Herwig [1] and Pythia [2]. The second is next-to-leading order (NLO) programs such as Nlojet++ [3] and Mcfm [4]. The NLO programs have NLO accuracy for hard cross sections like $\sigma[T_4]$, but do a bad job of predicting quantities like $d\sigma[T_4]/dM^2$ that represent the inner structure of jets. The parton shower Monte Carlo programs do not have NLO accuracy for hard cross sections but do a good job of predicting quantities like $d\sigma[T_4]/dM^2$. Additionally, they have the advantage that their final state particles are hadrons rather than partons. There are also some NLO-MC hybrids. One example is the program of Frixione, Nason, and Webber [5 − 7], which so far has been applied to cases with massless incoming partons but not to cases with massless final state partons at the Born level of calculation. The other example is that of [8 − 10], which concerns three-jet observables in electron-positron annihilation and thus addresses massless final state partons but not massless initial state partons. This paper concerns a method for constructing NLO-MC hybrid programs.

The shower Monte Carlo event generators can be made more powerful by incorporating the k_T-jet matching scheme of Catani, Kuhn, Krauss and Webber [11]. This scheme is defined by first considering the cross section $\sigma[F]$ to be divided into partial cross sections

$$\sigma[F] = \sum_{m=2}^{\infty} \sigma_m[F]\,. \tag{1.3}$$

Here $\sigma_m[F]$ consists of the contribution to $\sigma[F]$ from final states that are classified as consisting of m jets according to a slightly modified version of the k_T jet algorithm (as described in section 5 of this paper). This jet algorithm depends on the choice of a resolution scale for the jets, which we call $d_{\rm ini}$.

A parton shower Monte Carlo program based on the k_T-jet matching scheme is based on the Born squared matrix element for σ_m multiplied by a reweighting factor that we call W, which contains Sudakov factors of the sort found in parton shower Monte Carlo programs. This produces a realistic structure for jets at a scale above $d_{\rm ini}$. The base calculation for $\sigma_m[F]$ produces m partons. For each of these partons, the full program then produces a complete shower with $d_{\rm ini}$ as a maximum scale. This fills in the details of jet structure at scales below $d_{\rm ini}$. Both the factor W and the Sudakov factors that control the shower development at scales below $d_{\rm ini}$ depend on $d_{\rm ini}$, but the product of these two factors is approximately independent of $d_{\rm ini}$.

Suppose that we use a program of this sort to calculate $\sigma[F]$ for an infrared safe N-jet observable as defined in section 2. We assume that the definition F is such that the perturbative expansion of $\sigma[F]$ does not involve large logarithms. Thus F measures large momentum scale features of events with N or more jets. It vanishes if there are fewer than N narrow jets. As long as the jet resolution parameter $d_{\rm ini}$ is appropriately chosen, the main contribution to $\sigma[F]$ comes from $\sigma_m[F]$ with $m = N$. This contribution is of order $\alpha_{\rm s}^{B_m}$, where $B_m = m-2$ for $e^+e^- \to hadrons$. For instance, in our example, T_4 is a three jet observable. A calculation using PYTHIA with $\sqrt{s} = M_Z$ shows that if we pick $d_{\rm ini} \approx 0.03$, then $\sigma_3[T_4]$ gives the dominant contribution to $\sigma[T_4]$, with $\sigma_2[T_4] \approx \sigma_4[T_4] \approx 0.1\,\sigma_3[T_4]$. We discuss this further in section 5.

The perturbative expansion of $\sigma_m[F]$ has the form

$$\sigma_m[F] = C_{m,0}[F]\,\alpha_{\rm s}^{B_m}(Q) + C_{m,1}[F]\,\alpha_{\rm s}^{B_m+1}(Q) + C_{m,2}[F]\,\alpha_{\rm s}^{B_m+2}(Q) + \cdots . \tag{1.4}$$

(This applies for any m, not just $m = N$, but some of the coefficients are zero if $m < N$.) A leading order parton shower Monte Carlo program based on the k_T-jet matching scheme will get the Born term, $C_{m,0}[F]\,\alpha_{\rm s}^{B_m}(Q)$, in this expansion exactly right. Higher order contributions will be present, but getting the higher order perturbative contributions correctly is beyond the order of approximation intended in such a program.

If we wanted more perturbative accuracy than described above, we might use an NLO calculation. There are, in fact, NLO calculations available for a wide variety of important processes. Modern calculations are in the form of computer programs, each of which is designed to work for a certain class of observables F. Such programs are very important for producing accurate predictions for the class of measurement functions F for which they were designed. Unfortunately ordinary NLO programs have significant limitations. For instance, an NLO program can calculate $\sigma[T_4]$, but if one were to use the same program to calculate $d\sigma[T_4]/dM^2$ for $M^2/s \ll 1$, the result would not be even qualitatively right. For $\sqrt{s} = M_Z$, the physical distribution peaks at a jet mass of a few GeV and tends to zero for $M \to 0$. In contrast, the perturbative program gives for $d\sigma[T_4]/dM^2$ a result that increases without bound as $M^2 \to 0$ and additionally contains a delta function at zero mass, $\delta(M^2)$, with a coefficient that is negative and infinite (see [10] for a similar example worked out numerically).

The purpose of this paper is to extend to next-to-leading order the parton shower Monte Carlo idea based on the k_T-jet matching scheme. That is, we want to keep the feature that each outgoing hard parton generates a full parton shower and hadronization. We also want to utilize the decomposition of $\sigma[F]$ into $\sum \sigma_m[F]$ so that we can include calculations for different numbers of hard jets in the same program. However, we now want the perturbative expansion of the calculated $\sigma_m[F]$ to be correct with respect to the first *two* coefficients in eq. (1.4), namely $C_{m,0}[F]$ and $C_{m,1}[F]$.[1]

The general idea of the algorithm that we present applies, we believe, to lepton-lepton collisions, lepton-hadron collisions, and hadron-hadron collisions. This is because the dipole subtraction scheme that we use applies to all of these cases. However, we have elected to work out only the case of lepton-lepton collisions here and to leave the other cases for future publications.

Before we launch into a construction that must be rather involved if it is to be precise, it may be useful to begin with an informal preview. Consider as an example the calculation $\sigma[T_4]$. The dominant contribution to $\sigma[T_4]$ comes from $\sigma_3[T_4]$. This quantity has a perturbative expansion of the form (1.4) with $B_3 = 1$. We seek a calculation that includes parton

[1] Of course, we can have NLO accuracy only for those values of m for which NLO calculations exist. For $e^+e^- \to$ hadrons, that would currently be $m = 2, 3$, and 4.

showering and hadronization and gets the two coefficients $C_{3,0}[T_4]$ and $C_{3,1}[T_4]$ right. At the Born level ($\alpha_s^{B_3}$) there are three partons in the final state. At order $\alpha_s^{B_3+1}$ there can be four partons and we might think of the four partons as arising from three partons by parton splitting, say $q \to q+g$, even though the actual calculation involves the exact $\alpha_s^{B_3+1}$ matrix element. If we now add parton showering to one of the Born graphs we are in big trouble. The showering includes splittings $q \to q+g$, which we had already included. To include parton shower splittings of the Born level partons without double counting, we must expand their effect in perturbation theory and subtract the order $\alpha_s^{B_3+1}$ contributions from the $\alpha_s^{B_3+1}$ graphs.

We see that there are subtractions from the order $\alpha_s^{B_3+1}$ graphs that are derived from the splitting functions that generate the initial step of the parton shower. That is good, because NLO calculations inevitably involve subtractions. In this paper, we want to have an algorithm that can be used by NLO practitioners in a reasonably straightforward manner. For this reason, we design the parton splitting so that the subtractions are (with some minor changes) those of the dipole subtraction scheme of Catani and Seymour [12]. This scheme is expressed in a Lorentz covariant style and nicely expresses the complete available phase space for parton splitting. It is quite widely used for NLO calculations (for example in the programs NLOJET++ [3] and MCFM [4]).

The one parton splitting just described represents the hardest splitting in the eventual shower. After this comes secondary showering that can be performed according to an existing shower Monte Carlo program suitably modified to account for the starting condition that the hardest splitting has already occurred.

Now, the main contribution to a calculation of $\sigma[T_4]$ comes from final states that have three jets. The first splitting and the secondary showering generates the inner structure of these jets, which would be reflected in $d\sigma[T_4]/dM^2$. Experience has shown that quite a variety of showering algorithms are capable of doing a good job of representing this inner structure. In this paper, we leave open the choice of methods for the secondary showers. As long as the secondary showering has certain simple properties, the coefficients $C_{3,0}[T_4]$ and $C_{3,1}[T_4]$ will not be affected.

With this structure, we hope to largely decouple the part of the calculation that needs NLO computations from the bulk of the parton showering and hadronization. We have in mind that NLO practitioners could then write code that could be used in conjunction with a variety of Monte Carlo event generators, even with Monte Carlo event generators that are written after the NLO code.

We seek to be quite flexible not only with respect to the Monte Carlo event generator but also with respect to several of the function choices internal to the algorithm. The reason for this flexibility is to allow for improvements in the methods we suggest.

The complete algorithm calculates $\sigma[F]$ in the form

$$\sigma[F] \approx \frac{1}{N}\sum_{n=1}^{N} w_n\, F(f_n)\,. \tag{1.5}$$

That is, it is in the form of a Monte Carlo event generator with weights w_n. As is natural for a description that includes quantum interference, the weights for individual simulated events can be positive or negative.[2]

The algorithm presented in this paper combines two separate ideas. First, we use the k_T-jet matching scheme to match calculations involving different numbers of hard jets [11]. One might call this m-jet/$(m+1)$-jet matching. Second, we use the dipole subtraction scheme [12] to match between the hardest splitting in a parton shower and the next-to-leading order contributions to the m-jet matrix elements. One might call this PS/NLO matching. Most of the paper concerns PS/NLO matching. The reader might want to work on this part first. For this purpose, one can simply take $d_{\rm ini}$ to be infinitesimally less than 1. In that case, the partial cross sections $\sigma_m[F]$ all vanish except for $\sigma_2[F]$ and the reweighting factors W that appear in the paper can be taken to be $W = 1$. Of course such a calculation would be of only limited phenomenological usefulness since it would apply only to two-jet infrared-safe observables F. To get to a method that applies to three-jet observables, we need to do something about the two-jet region and then we need m-jet/$(m+1)$-jet matching.

In the following sections, we review the dipole algorithm [12] for doing NLO calculations in the case of electron-positron annihilation. For readers who are familiar with this algorithm, our aim is to set up the notation that we use, which differs in some instances from that used in ref. [12]. For other readers, our aim is to provide a compact introduction to the algorithm. We provide no proofs that the algorithm works. Furthermore, there are several functions that must be defined in order to fully specify the algorithm, but in the introductory sections that follow, we skip the formulas for these functions and merely state the important properties of the functions that follow from these formulas. The formulas are then summarized later on, in sections 10, 11 and 12. Once the dipole subtraction formalism has been set up, we define in section 5 how to break up the cross section into partial cross sections involving different numbers of resolved jets, along the lines of ref. [11]. Then in section 6 we modify the subtraction scheme just a little to accommodate this division. The main subject of the paper begins in section 7 with an outline of the general structure for adding showers to the partial cross sections. This is followed by an exposition in section 8, with several subsections, of how the showers are to be added while keeping track of the next-to-leading order correction terms needed to keep from changing the coefficient $C_{m,1}[F]$ in eq. (1.4). The following sections contain details on the functions used in the dipole subtraction and showering constructions. We present some conclusions in section 16.

2. Notations

In this section, we introduce some of the notations that we will use throughout the paper.

We wish to describe the process $e^+ + e^- \to$ *hadrons*. In a calculation at a finite order of perturbation theory, we consider final states consisting of n partons, with $n \geq 2$. We

[2] In NLO programs, there are also weights w_n, but the weights can be very large, with an event with a large positive weight being followed by a counter-event with a large negative weight. Here, the weights are not large.

denote the momenta of these partons by $\{p\}_n = \{p_1, p_2, \ldots, p_n\}$. We represent the phase space integration for the final state partons as

$$d\Gamma(\{p\}_n) = \prod_{l=1}^{n}\Big((2\pi)^{-3} d^4 p_l\, \delta_+(p_l^2)\Big)(2\pi)^4 \delta^4\left(\sum_{l=1}^{n} p_l - P_0\right), \tag{2.1}$$

where P_0 is the momentum of the initial e^+e^- pair, $P_0 = (\sqrt{s}, 0, 0, 0)$ if we use the c.m. frame.

We denote the flavors of the partons by labels $\{f\}_n$ with $f_i \in \{\mathrm{g}, \mathrm{u}, \bar{\mathrm{u}}, \mathrm{d}, \bar{\mathrm{d}}, \ldots\}$. Then the complete description of the final state momenta and flavors is specified by the list $\{p, f\}_n = \{(p_1, f_1), (p_2, f_2), \ldots, (p_n, f_n)\}$. We can define "addition" on flavors by saying that $f_1 + f_2 = f_3$ if there is a QCD vertex for $f_1 + f_2 \to f_3$. Thus, for instance, $\bar{\mathrm{d}} + \mathrm{g} = \bar{\mathrm{d}}$ and $\mathrm{u} + \bar{\mathrm{u}} = \mathrm{g}$, while $\mathrm{d} + \bar{\mathrm{u}}$ is not defined. The splitting of a final state parton with flavor f_l can be represented by giving the pair of daughter flavors $\{\hat{f}_{l,1}, \hat{f}_{l,2}\}$ in the set of flavor pairs such that $\hat{f}_{l,1} + \hat{f}_{l,2} = f_l$.

The matrix element $\mathcal{M}$ for $a + b \to n$ *partons* depends on the spin and color indices of the n partons. In order to avoid writing these indices, we follow the notation of Catani and Seymour and write the matrix element as a vector

$$\big|\mathcal{M}(\{p, f\}_n)\big\rangle \tag{2.2}$$

in color $\otimes$ spin space. The squared matrix element, summed over colors and spins, is then

$$\big\langle\mathcal{M}(\{p, f\}_n)\big|\mathcal{M}(\{p, f\}_n)\big\rangle\,. \tag{2.3}$$

We consider a perturbative calculation that is designed to predict an infrared safe N-jet observable. We need to be careful about what we mean by an infrared safe observable. There should be a function $F_n(\{p_1, p_2, \ldots, p_n\})$ defined for any number n of massless partons with momenta p_i. The functions F_n do not depend on the flavors of the partons and should be invariant under permutations of the momentum arguments $p_1, p_2, \ldots, p_n$. In a calculation at "all" orders of perturbation theory, the cross section for the observable would be

$$\sigma[F] = \sum_n \frac{1}{n!}\int d\vec{p}_1 \cdots d\vec{p}_n\, \frac{d\sigma}{d\vec{p}_1 \cdots d\vec{p}_n}\, F_n(\{p\}_n)\,. \tag{2.4}$$

Here $p_i^0 = |\vec{p}_i|$ in F and $d\sigma$ contains the delta functions for momentum and energy conservation. The cross section needs to contain a regulator to control infrared divergences, which cancel between terms with different numbers n of partons. To ensure that these cancellations work, the F_n functions for different values of n need to be related. Specifically, if two partons become collinear or one becomes soft ($p_i \to 0$), the outcome of the measurement should be unaffected:

$$F_n(\{p_1, p_2, \ldots, (1-\lambda)p_{n-1}, \lambda p_{n-1}\}) = F_{n-1}(\{p_1, p_2, \ldots, p_{n-1}\}) \tag{2.5}$$

for $0 \le \lambda < 1$. (The case $\lambda = 0$ covers the possibility that parton n is soft and not necessarily collinear with any other parton.) Cancellation of infrared divergences also requires that the measurement is unaffected when many partons become soft or group themselves into collinear groups (jets). To cover this possibility, we also require that the functions $F_n(\{p\}_n)$ be smooth functions of their arguments.[3]

Ultimately, one wants to apply the measurement function to hadrons. Indeed, we want to do that in this paper after supplementing the perturbative calculation with a simulation of parton showering and hadronization. For this purpose, one uses the same functions F_n, but now applied to hadron momenta p_i with $p_i^2 > 0$. The extensions of the F_n to $p_i^2 > 0$ should then have smooth limits as any or all of the p_i^2 approach 0.

We have defined what we mean by an infrared safe observable. In this paper, we consider an infrared-safe "N-jet" observable. This means that additionally

$$F_m(\{p_1, p_2, \dots, p_m\}) = 0\,, \qquad m < N\,. \tag{2.6}$$

Throughout the paper we use the standard notations $C_{\rm F} = (N_{\rm c}^2 - 1)/2N_{\rm c}$, $C_{\rm A} = N_{\rm c}$, $T_{\rm R} = 1/2$ for color SU($N_{\rm c}$). We assume that there are $N_{\rm f}$ flavors (+ $N_{\rm f}$ antiflavors) of quarks in the fundamental representation. We typically use the notation f to represent a parton flavor, $f \in \{{\rm g}, {\rm u}, \bar{\rm u}, {\rm d}, \dots\}$. Then we use coefficients C_f, K_f, and γ_f defined by

$$\begin{aligned}
C_f &= C_{\rm F} \qquad \text{for } f \in \{{\rm u}, \bar{\rm u}, {\rm d}, \dots\}\,,\\
C_{\rm g} &= C_{\rm A}\,,\\
\gamma_f &= \frac{3}{2}\,C_{\rm F} \qquad \text{for } f \in \{{\rm u}, \bar{\rm u}, {\rm d}, \dots\}\,,\\
\gamma_{\rm g} &= \frac{11}{6}C_{\rm A} - \frac{2}{3}T_{\rm R}N_{\rm f}\,,\\
K_f &= \left(\frac{7}{2} - \frac{\pi^2}{6}\right)C_{\rm F} \qquad \text{for } f \in \{{\rm u}, \bar{\rm u}, {\rm d}, \dots\}\,,\\
K_{\rm g} &= \left(\frac{67}{18} - \frac{\pi^2}{6}\right)C_{\rm A} - \frac{10}{9}T_{\rm R}N_{\rm f}\,.
\end{aligned} \tag{2.7}$$

The running coupling is $\alpha_{\rm s}(\mu)$ evaluated at scale μ, often a transverse momentum. When no scale is indicated, we mean $\alpha_{\rm s} \equiv \alpha_{\rm s}(\mu_{\rm R})$ where the $\mu_{\rm R}$ is a fixed renormalization scale, usually chosen as some fraction of $\sqrt{s}$. For our NLO calculations, we would use the two loop running coupling. In construction our matching with parton showers, we will require the first order relation

$$\frac{\alpha_{\rm s}(\mu)}{\alpha_{\rm s}(\mu_{\rm R})} = 1 - \beta_0 \log\left(\frac{\mu^2}{\mu_{\rm R}^2}\right)\frac{\alpha_{\rm s}(\mu_{\rm R})}{2\pi} + \mathcal{O}(\alpha_{\rm s}^2)\,, \tag{2.8}$$

where $\beta_0 = \gamma_{\rm g}$.

3. Construction and deconstruction of parton splitting

The dipole algorithm of Catani and Seymour [12] is based on a physical picture involving parton splitting, which turns m partons into $m + 1$ partons. Deconstructing the splitting

[3] A smooth function has an infinite number of derivatives, and thus a Taylor expansion to any order. This property is stronger than one really needs, but successively weaker requirements become successively more unwieldy.

turns the $m+1$ partons back into m partons. In this section, we describe the splitting kinematics without giving the detailed formulas from ref. [12], which are given in section 10.

We describe deconstruction first. Suppose that we have a list of $m+1$ parton momenta and flavors, $\{\hat{p},\hat{f}\}_{m+1}$. One imagines that partons i and j are produced by the splitting of a mother parton with flavor $\tilde{f}_{ij}$ and momentum $\tilde{p}_{ij}$. We need one more parton, with index k, to describe the splitting in the scheme of Catani and Seymour. Parton k is a "spectator parton" that absorbs some momentum associated with the splitting.[4]

Consider first the flavors. The mother parton has flavor $\tilde{f}_{ij} = \hat{f}_i + \hat{f}_j$. The flavor of the spectator parton is not changed: $\tilde{f}_k = \hat{f}_k$. There is less information in the list of just one flavor $\tilde{f}_{ij}$ than in the list of two flavors $\{\hat{f}_i,\hat{f}_j\}$. The missing information is the flavor splitting choice, which can be specified by giving the pair $\{\hat{f}_i,\hat{f}_j\}$ in the set with $\hat{f}_i + \hat{f}_j = \tilde{f}_{ij}$.

We now extend this idea to the momenta. The massless momenta $\{\hat{p}_i,\hat{p}_j,\hat{p}_k\}$ determine new momenta $\{\tilde{p}_{ij},\tilde{p}_k\}$ of just two on-shell massless partons together with three splitting variables. The structure of this transformation is simple in the limiting case of collinear splitting. If $\hat{p}_i\cdot\hat{p}_j = 0$, we have

$$\tilde{p}_{ij} = \hat{p}_i + \hat{p}_j \qquad (\text{when } \hat{p}_i\cdot\hat{p}_j = 0)\,, \tag{3.1}$$

while the momentum of the spectator parton remains unchanged,

$$\tilde{p}_k = \hat{p}_k \qquad (\text{when } \hat{p}_i\cdot\hat{p}_j = 0)\,. \tag{3.2}$$

It is not possible to retain these relations away from the collinear limit. However, Catani and Seymour still maintain

$$\tilde{p}_{ij} + \tilde{p}_k = \hat{p}_i + \hat{p}_j + \hat{p}_k \tag{3.3}$$

while keeping all of the momenta massless.

Since one eliminates three degrees of freedom in going from $\{\hat{p}_i,\hat{p}_j,\hat{p}_k\}$ to $\{\tilde{p}_{ij},\tilde{p}_k\}$, we can supplement $\{\tilde{p}_{ij},\tilde{p}_k\}$ with three splitting variables. For our purposes, it is convenient to call these y,z,ϕ. The most important role in the formalism is played by the dimensionless variable y proportional to the virtuality of the splitting, so that

$$y = 0 \qquad \text{if and only if } \hat{p}_i\cdot\hat{p}_j = 0\,. \tag{3.4}$$

The variable z is a momentum fraction representing the share of the mother parton momentum that is carried by parton i and ϕ is an azimuthal angle.

The discussion above can be summarized by saying that there is a map

$$\{(\hat{p}_i,\hat{f}_i),(\hat{p}_j,\hat{f}_j),(\hat{p}_k,\hat{f}_k)\} \to \{(\tilde{p}_{ij},\tilde{f}_{ij}),(\tilde{p}_k,\tilde{f}_k),y,z,\phi,\hat{f}_i,\hat{f}_j\}\,. \tag{3.5}$$

Parton splitting is the other way around. Here we begin with a list of the momenta and flavors $\{p,f\}_m$ of m partons. We imagine that one of these, parton l, splits, producing

[4]The spectator parton also plays a role in the color flow associated with the splitting, as expressed in the color matrices in the definition of $\mathcal{D}_{ij,k}$ below.

daughter partons with with flavors $\{\hat{f}_{l,1}, \hat{f}_{l,2}\}$ and momenta $\{\hat{p}_{l,1}, \hat{p}_{l,2}\}$. We again need a spectator parton, with index k. To specify the splitting we need splitting parameters $\{y, z, \phi, \hat{f}_{l,1}, \hat{f}_{l,2}\}$, with $\hat{f}_{l,1} + \hat{f}_{l,2} = f_l$. Then Catani and Seymour specify a map

$$\{(p_l, f_l), (p_k, f_k), y, z, \phi, \hat{f}_{l,1}, \hat{f}_{l,2}\} \to \{(\hat{p}_{l,1}, \hat{f}_{l,1}), (\hat{p}_{l,2}, \hat{f}_{l,2}), (\hat{p}_k, \hat{f}_k)\}\,. \tag{3.6}$$

The map (3.6) is precisely the inverse of the map (3.5). Only the variable names are different. The part of these maps that concerns the flavor splitting is trivial. The part that concerns the momentum splitting is not trivial. The formulas from ref. [12] are given in section 10.

These maps can be rather trivially extended to include all of the partons. Suppose that we start with a list $\{p, f\}_m$ of m parton momenta and flavors and that we want to split parton l with the help of spectator parton k using splitting variables $\{y, z, \phi, \hat{f}_{l,1}, \hat{f}_{l,2}\}$. We need $\hat{f}_{l,1} + \hat{f}_{l,2} = f_l$. We can produce a list $\{\hat{p}, \hat{f}\}_{m+1}$ of $m+1$ parton momenta and flavors by removing partons l and k from the original list and adding $\hat{p}_{l,1}, \hat{f}_{l,1}, \hat{p}_{l,2}, \hat{f}_{l,2}$ and $\hat{p}_k, \hat{f}_k$ from eq. (3.6) to the end of the list. Then $\{\hat{p}, \hat{f}\}_{m+1}$ equals the new list. It will prove useful to call the complete transformation $\mathcal{R}_{l,k}$,

$$\{\hat{p}, \hat{f}\}_{m+1} = \mathcal{R}_{l,k}(\{p, f\}_m, \{y, z, \phi, \hat{f}_{l,1}, \hat{f}_{l,2}\})\,. \tag{3.7}$$

Now suppose that we start with a list $\{\hat{p}, \hat{f}\}_{m+1}$ of $m+1$ parton momenta and flavors and that we want to combine partons i and j with the help of spectator parton k. We can produce a list $\{p, f\}_m$ of m parton momenta and flavors by removing partons i, j and k from the original list and adding $\tilde{p}_{ij}, \tilde{f}_{ij}$ and $\tilde{p}_k, \tilde{f}_k$ from eq. (3.5) to the end of the list. Then $\{p, f\}_m$ equals the new list. We also get the splitting variables $\{y, z, \phi, \hat{f}_i, \hat{f}_j\}$ with the help of eq. (3.5). It will prove useful to call the complete transformation $\mathcal{Q}_{ij,k}$,

$$\{\{p, f\}_m, \{y, z, \phi, \hat{f}_i, \hat{f}_j\}\} = \mathcal{Q}_{ij,k}(\{\hat{p}, \hat{f}\}_{m+1})\,. \tag{3.8}$$

The transformations $\mathcal{R}$ and $\mathcal{Q}$ are inverses of each other in the sense that if we supply the right permutation operators $\Pi(i, j, k)$ and $\Pi(l, k')$ we have $\Pi(i, j, k)\mathcal{R}_{l,k'}\Pi(l, k')\mathcal{Q}_{ij,k} = 1$. The permutations will not be of much concern to us since the functions that we use that are functions of $\{p, f\}$ are invariant under permutations of the parton labels.

Where possible, for a final state parton l, we denote the complete set of splitting variables by a single letter,

$$Y_l = \{y_l, z_l, \phi_l, \hat{f}_{l,1}, \hat{f}_{l,2}\}\,. \tag{3.9}$$

With this notation we can abbreviate

$$\int_0^1 \frac{dy_l}{y_l} \int_0^1 dz_l \int_0^{2\pi} \frac{d\phi_l}{2\pi}\, \frac{1}{2} \sum_{\hat{f}_{l,1}, \hat{f}_{l,2}} \delta^{f_l}_{\hat{f}_{l,1}+\hat{f}_{l,2}} \equiv \int dY_l\,. \tag{3.10}$$

Now that we understand the parton splitting, we are ready to examine the construction of the cross section at NLO.

4. Structure of the NLO cross section

We consider an N-jet cross section in electron-positron collisions. The cross section correct to NLO is based on a tree level cross section for the production of m partons, with $m = N$, together with one loop graphs for the production of m partons and tree graphs for the production of $m+1$ partons. The cross section is constructed as a sum of terms

$$\sigma^{\rm NLO} = \sigma^{\rm B} + \sigma^{\rm R-A} + \sigma^{\rm V+A}\,. \tag{4.1}$$

In the first term there is an integration over m-parton phase space, while in the second term there is an integration over $m+1$-parton phase space. The first term is the Born contribution, proportional to $\alpha_{\rm s}^{B_m} \equiv \alpha_{\rm s}^{m-2}$. The second term is a correction proportional to $\alpha_{\rm s}^{B_m+1}$ associated with real parton emission, which comes with a subtraction term that eliminates the soft and collinear divergences. The third term is a correction proportional to $\alpha_{\rm s}^{B_m+1}$ that is associated with a virtual parton loop. There are m partons in the final state. There is a corresponding subtraction term in which there is an integration over the phase space for one parton, which is performed analytically in $4-2\epsilon$ dimensions to produce $1/\epsilon^2$ and $1/\epsilon$ terms that cancel $1/\epsilon^2$ and $1/\epsilon$ terms that would be present without the subtraction.

We will begin with the Born contribution.

4.1 The Born contribution

The Born contribution takes the form

$$\sigma^{\rm B} = \frac{1}{m!}\sum_{\{f\}_m}\int d\Gamma(\{p\}_m)\big\langle\mathcal{M}(\{p,f\}_m)\big|\mathcal{M}(\{p,f\}_m)\big\rangle\,F_m(\{p\}_m)\,. \tag{4.2}$$

There is an integration over the final state momenta and a sum over final state flavors $\{f\}_m$ with a symmetry factor $1/m!$. Next is the squared matrix element for the production of the m final state partons. Finally, there is a final state measurement function, $F_m(\{p\}_m)$.

4.2 The real emission contribution

The real emission contribution $\int d\sigma^{\rm R}$ along with its subtraction $\int d\sigma^{\rm A}$ has the form

$$\begin{aligned}
\sigma^{\rm R-A} &= \int_{m+1}\left[d\sigma^{\rm R} - d\sigma^{\rm A}\right] \\
&= \frac{1}{(m+1)!}\sum_{\{\hat f\}_{m+1}}\int d\Gamma(\{\hat p\}_{m+1})\,\times \\
&\quad\times\Bigg[\big\langle\mathcal{M}(\{\hat p,\hat f\}_{m+1})\big|\mathcal{M}(\{\hat p,\hat f\}_{m+1})\big\rangle\,F_{m+1}(\{\hat p\}_{m+1}) - \\
&\quad - \sum_{\substack{i,j\\ \text{pairs}}}\sum_{k\neq i,j}\mathcal{D}_{ij,k}(\{\hat p,\hat f\}_{m+1})\,F_m(\{p\}_m^{ij,k})\Bigg]\,.
\end{aligned} \tag{4.3}$$

The real emission term $d\sigma^{\rm R}$ is represented by the first term in square brackets. Here we have the same sums and integrals as in $\sigma^{\rm B}$ except that now there are $m+1$ final state partons. The momentum and flavor variables are all written with hats in order to distinguish them from the m-parton variables. There is the squared amplitude to produce the $m+1$ partons. The resulting partonic cross section is multiplied by the measurement function F for $m+1$ partons.

The contribution $d\sigma^{\rm R}$ has a potential singularity when any of the dot products $\hat p_i\cdot\hat p_j$ tends to zero. In the second term in the square brackets, representing $d\sigma^{\rm A}$, we sum over subtractions designed to reduce the strength of these singularities. The subtractions are labeled by (unordered) pairs $\{i,j\}$ where $i,j\in\{1,\dots,m+1\}$. For each choice of $\{i,j\}$, there are a number of subtraction terms labeled by the index k of a spectator parton, which can be any of the partons except i or j. We use the splitting deconstruction transformation from eq. (3.8) to define m parton variables and splitting variables according to

$$\{\{p,f\}_m^{ij,k}, Y_{ij,k}\} = \mathcal{Q}_{ij,k}(\{\hat p,\hat f\}_{m+1})\,. \tag{4.4}$$

The ij,k superscripts or subscripts remind us that the definition of these variables is different for each choice of parton indices i,j,k.

The subtraction term in eq. (4.3) contains a dipole subtraction function $\mathcal{D}_{ij,k}$. We state the definitions in section 11, eqs. (11.2), (11.3), (11.4), (11.5), (11.6), (11.7), but here simply note that $\mathcal{D}_{ij,k}$ has the structure

$$\begin{aligned}&\mathcal{D}_{ij,k}(\{\hat p,\hat f\}_{m+1}) =\\ &\quad= \frac{1}{2\hat p_i\cdot\hat p_j}\,\big\langle \mathcal{M}(\{p,f\}_m^{ij,k})\big|\frac{\boldsymbol{T}_{ij}\cdot\boldsymbol{T}_k}{-\boldsymbol{T}_{ij}^2}\,\boldsymbol{V}_{ij}(\{\tilde p_{ij},y,z,\phi\}_{ij,k},f_i,f_j)\,\big|\mathcal{M}(\{p,f\}_m^{ij,k})\big\rangle\,.\end{aligned} \tag{4.5}$$

The function $\mathcal{D}_{ij,k}$ is based on the Born level amplitude $|\mathcal{M}(\{p,f\}_m^{ij,k})\rangle$ for the flavors and momenta with partons i and j combined. There is an operator $\boldsymbol{T}_{ij}\cdot\boldsymbol{T}_k/\boldsymbol{T}_{ij}^2$ that acts on the color degrees of freedom of $|\mathcal{M}\rangle$. Specifically, $\boldsymbol{T}_{ij}^a$ is the infinitesimal SU(3) generator matrix in the a direction acting on the color indices for the mother parton that results from combining partons i and j (which, to be precise, has been placed in the second-to-last parton slot by the transformation $\mathcal{Q}_{ij,k}$). Similarly, $\boldsymbol{T}_k^a$ is the generator matrix acting on the color indices for the spectator parton. Then $\boldsymbol{T}_{ij}\cdot\boldsymbol{T}_k\equiv\sum_a\boldsymbol{T}_{ij}^a\boldsymbol{T}_k^a$. In the denominator, $\boldsymbol{T}_{ij}^2\equiv\sum_a\boldsymbol{T}_{ij}^a\boldsymbol{T}_{ij}^a$ is a number, $C_{\rm F}$ or $C_{\rm A}$, depending on whether parton ij is quark or antiquark or a gluon. There is an important identity involving the color matrices. Invariance of $\big|\mathcal{M}\big\rangle$ under color rotations implies that $(\boldsymbol{T}_{ij}+\sum_k\boldsymbol{T}_k)\big|\mathcal{M}\big\rangle=0$. Thus

$$\sum_{k\neq ij}\frac{\boldsymbol{T}_{ij}\cdot\boldsymbol{T}_k}{-\boldsymbol{T}_{ij}^2} = 1 \tag{4.6}$$

when operating on $\big|\mathcal{M}\big\rangle$. Next in eq. (4.5), there is an operator $\boldsymbol{V}_{ij,k}$ that depends on the momenta and acts on the spin degrees of freedom of $|\mathcal{M}\rangle$. The dipole subtraction functions $\mathcal{D}_{ij,k}$ are constructed so that their sum matches the squared matrix element $\big\langle\mathcal{M}(\{\hat p,\hat f\}_{m+1})\big|\mathcal{M}(\{\hat p,\hat f\}_{m+1})\big\rangle$ in the limit in which the matrix element is singular.

Taking some liberties with the notation, we will also use $\mathcal{D}_{l,k}(\{p,f\}_m;Y)$ to denote the same function written in terms of the variables $\{\{p,f\}_m;Y\}$ given by eq. (3.8).

Next in $d\hat{\sigma}^{\rm A}$ there is a measurement function $F_J(\{p\}_m^{ij,k})$ evaluated at the momenta for the m-parton state. The two measurement functions in the two terms in eq. (4.3) match in the limits in which the matrix element is singular because of the infrared safety property (2.5).

4.3 The virtual loop contribution

The virtual loop contribution along with its counterterm has the form

$$\sigma^{\rm V+A} = \int_m \Big[d\sigma^{\rm V} + \int_1 d\sigma^{\rm A}\Big] = \frac{1}{m!}\sum_{\{f\}_m}\int d\Gamma(\{p\}_m)\, V(\{p,f\}_m)\, F_m(\{p\}_m)\,. \tag{4.7}$$

The function V comes from the one loop matrix element. The matrix element has been calculated analytically in $4-2\epsilon$ dimensions. Then its $1/\epsilon^n$ pole terms and certain finite pieces have been subtracted. The result V is constructed from Born amplitude and certain spin and flavor dependent functions. The terms subtracted are precisely what was subtracted from the real emission contribution, but with the opposite sign. The details of V are laid out in section 12, eqs. (12.2), (12.3), (12.4).

5. Partial cross sections

In subsequent sections, we will add parton showers to our calculation. Before we do this, however, it is useful to divide the cross section into partial cross sections σ_m with $m = 2,3,\dots$ that are based on the cross sections for 2 partons $\to m$ partons and have higher order corrections added. This construction is based on the k_T-jet matching scheme of Catani, Krauss, Kuhn, and Webber [11].

We consider the calculation of an infrared safe N-jet observable, as described in section 2. In particular, the measurement function F obeys eq. (2.6), $F_m(\{p_1,p_2,\dots,p_m\}) = 0$ for $m < N$. We organize this calculation according to the number of partons in the final state,

$$\sigma[F] = \sum_{n=N}^{\infty}\frac{1}{n!}\sum_{\{f\}_n}\int d\Gamma(\{p\}_n)\, G_n(\{p,f\}_n)\, F_n(\{p\}_n)\,. \tag{5.1}$$

Here G_n has a lowest order contribution proportional to $\alpha_{\rm s}^{B_n}$, where $B_n = n-2$,

$$G_n^0 = \big\langle \mathcal{M}(\{p,f\}_n)\big|\mathcal{M}(\{p,f\}_n)\big\rangle\,. \tag{5.2}$$

There are higher order contributions, proportional to $\alpha_{\rm s}^{B_n+L}$ with L virtual loops. When the cross section is expanded in this form, some kind of regulation is needed on the integrals; the divergences then cancel between terms with different numbers of partons n.

Given an n-parton final state, $\{p,f\}_n$, we group the partons into jets. We use the recursive "k_T" jet finding algorithm [13], modified slightly to make use of the parton flavor information available. At each stage of the algorithm there is a dimensionless jet resolution function d_{ij} that approximates (for small angles and virtualities) the squared transverse momentum of one jet (group of partons or a single parton) with label i with respect to another with label j with which it might be grouped, divided by s:

$$d_{ij} \equiv d(p_i, p_j) = \frac{2}{s} \min\left[E_i^2, E_j^2\right](1 - \cos(\theta_{ij}))\,. \tag{5.3}$$

Here we use the e^+e^- c.m. frame to define θ_{ij} as the angle between $\vec{p}_i$ and $\vec{p}_j$ and to define $E_i = p_i^0$, $E_j = p_j^0$.

We begin with n partons. At each stage of the algorithm, we combine two partons, so that as the algorithm progresses, there are fewer and fewer partons left. There are some rules. In order to combine partons with flavors f_b and f_c, there must be a flavor f_a such that there is a QCD vertex for $f_a \to f_b f_c$. In addition, the initial splitting in $e^+e^- \to$ *hadrons* must be $\gamma/Z \to \bar{q}q$. Therefore, when working backwards from the final state, it is not allowed to combine the last remaining q-$\bar{q}$ pair into a gluon. Thus at each stage, we find all pairs $\{i, j\}$ of partons that are allowed to be combined by these flavor considerations. For each such pair, we calculate the distance measure d_{ij} given in eq. (5.3). The allowed pair with the smallest d_{ij} is combined by adding the four-momenta of the daughter partons to form the four-momentum of the mother and determining the flavor of the mother such that there is a QCD vertex for the mother to split to form the daughters.

This process gives a sequence of resolution parameters $d_J(\{p, f\}_n)$ at which two jets were joined, reducing J jets to $J-1$ jets. Typically $d_J < d_{J-1}$. However, the flavor considerations discussed above may invalidate this ordering when we must reject the parton pair with the smallest d_{ij} and choose a pair to combine that has a larger d_{ij}. This misordering does not happen in the leading approximation in a parton shower. If d_{J-1} calculated according to eq. (5.3) is smaller than d_J, we simply redefine d_{J-1} to equal d_J. Then the definition gives $d_n \le d_{n-1} \le \cdots \le d_3$. We also define $d_J(\{p, f\}_n) = 0$ for $J > n$.

We can use this k_T jet algorithm to divide the cross section into partial cross sections σ_m with specified integration ranges,

$$\sigma[F] = \sum_{m=2}^{\infty} \sigma_m[F]\,, \tag{5.4}$$

with

$$\begin{aligned} \sigma_m[F] = &\sum_{n=N}^{\infty} \frac{1}{n!} \sum_{\{f\}_n} \int d\Gamma(\{p\}_n)\, \theta(d_{m+1}(\{p, f\}_n) < d_{\rm ini} < d_m(\{p, f\}_n)) \times \\ &\times G_n(\{p, f\}_n)\, F_n(\{p\}_n)\,. \end{aligned} \tag{5.5}$$

In σ_m there are exactly m jets that are resolvable at a scale $d_{\rm ini}$ that can be chosen to suit our purposes, as discussed briefly in section 1. Although there may be more than m partons, there are not more resolvable jets. There are no infrared divergences in σ_m arising from two of the m jets becoming collinear or one of them becoming soft because the singular region is removed by the cut $d_{\rm ini} < d_m(\{p, f\}_n)$. There are also no infrared divergences in σ_m arising from the possible subjets becoming collinear or soft because of the cancellation between real and virtual graphs.

Assume that F is an infrared-safe N-jet observable that is sensitive only to event structure at a large momentum scale $Q \sim \sqrt{s}$. Then as long as the jet resolution parameter $d_{\rm ini}$ is appropriately chosen, the main contribution to $\sigma[F]$ comes from $\sigma_N[F]$, with N jets resolvable at scale $d_{\rm ini}$. We touched on this topic in section 1 and discuss it further in section 14.

Our aim in this paper will be to add showers to the perturbative calculation of $\sigma_m[F]$ in such a way that the first two terms in the perturbative expansion of $\sigma_m[F]$ are reproduced without "double counting" between the shower splittings and the splittings that are part of the contributions to $\sigma_m[F]$ of order $\alpha_{\rm s}^{B_m+1}$. First, however, we need to indicate how the dipole subtraction scheme for NLO perturbative calculations can be modified to work with the partial cross sections $\sigma_m[F]$,

6. Partial cross sections with dipole subtractions

The integrals in eq. (5.5) still need regulation because the real-virtual cancellations happen between terms with different values of n. However, as long as we limit the calculation to next-to-leading order, it is simple to adapt the dipole subtraction scheme to make the cancellations happen inside of each of the integrations. The idea is to write each partial cross section σ_m as a sum of three terms, with corrections suppressed by two powers of $\alpha_{\rm s}$,

$$\sigma_m = \sigma_m^{\rm B} + \sigma_m^{\rm R-A} + \sigma_m^{\rm V+A} + \mathcal{O}(\alpha_{\rm s}^{B_m+2})\,. \tag{6.1}$$

For the Born contribution, we simply modify eq. (4.2) by inserting the appropriate cut,

$$\begin{aligned}\sigma_m^{\rm B} = \frac{1}{m!}\sum_{\{f\}_m}\int d\Gamma(\{p\}_m)\,\theta(d_{\rm ini} < d_m(\{p,f\}_m))\times\\ \times\big\langle\mathcal{M}(\{p,f\}_m)\big|\mathcal{M}(\{p,f\}_m)\big\rangle\,F_m(\{p\}_m)\,.\end{aligned} \tag{6.2}$$

(Note that there is no cut $d_{m+1} < d_{\rm ini}$ since there are only m partons.)

In the real emission contribution, we follow eq. (4.3), inserting the $d_{m+1} < d_{\rm ini} < d_m$ cut in the main term and a similar cut in the subtraction term,

$$\begin{aligned}\sigma_m^{\rm R-A} = {}& \frac{1}{(m+1)!}\sum_{\{\hat f\}_{m+1}}\int d\Gamma(\{\hat p\}_{m+1})\times\\ &\times\Bigg[\big\langle\mathcal{M}(\{\hat p,\hat f\}_{m+1})\big|\mathcal{M}(\{\hat p,\hat f\}_{m+1})\big\rangle\,F_{m+1}(\{\hat p\}_{m+1})\times\\ &\quad\times\theta(d_{m+1}(\{\hat p,\hat f\}_{m+1}) < d_{\rm ini} < d_m(\{\hat p,\hat f\}_{m+1}))-\\ &\quad-\sum_{\substack{i,j\\ \text{pairs}}}\sum_{k\neq i,j}\mathcal{D}_{ij,k}(\{\hat p,\hat f\}_{m+1})\,F_m(\{p\}_m^{ij,k})\times\\ &\quad\times\theta\Big(\tilde d(\{p,f\}_m^{ij,k},\{l,y,z\}_{ij,k}) < d_{\rm ini} < d_m(\{p,f\}_m^{ij,k})\Big)\Bigg]\,.\end{aligned} \tag{6.3}$$

In the subtraction term, there is a sum over parton pairs $\{ij\}$ and spectator partons k. For each term, there is an appropriate cut on the momenta. First, the m-parton state that results from combining partons i and j must be resolvable at scale $d_{\rm ini}$. Second, the splitting of the mother parton thus obtained must be unresolvable according to a resolution function $\tilde{d}$,

$$\tilde{d}(\{p,f\}_m, l, y_l, z_l) = \frac{s_l}{s}\, y_l \min\left\{\frac{1-z_l}{z_l}, \frac{z_l}{1-z_l}\right\}. \tag{6.4}$$

Here s_l is a virtuality scale appropriate to parton l. The simplest choice would be $s_l = s$. However, for our purposes, it will prove useful to choose a value obtained from $\{p,f\}_m$ that we will specify in section 13, eq. (13.16). Aside from the factor s_l/s, the function $\tilde{d}(\{p,f\}_m, l, y_l, z_l)$ is derived from the resolution variable of the k_T jet finding algorithm, eq. (5.3). Except for a y and z independent factor, it gives an approximate version of the resolution variable associated with the splitting $\{p\}_m \to \{\hat{p}\}_{m+1}$ generated with the splitting parameters y_l, z_l with the help of spectator parton k_l,

$$d(\hat{p}_{l,1}, \hat{p}_{l,2}) = \frac{2p_l \cdot p_{k_l}}{s_l}\, \tilde{d}(\{p,f\}_m, l, y_l, z_l)\, [1 + \mathcal{O}(\sqrt{y_l})]\,. \tag{6.5}$$

We describe this more precisely in section 10 at eq. (10.19). The effect of the cuts on the dipole terms is easiest to understand in the case that for the $m+1$-parton state that we start with, one pair has a very small resolution parameter, while once that pair is combined the other pairs are well separated. In this case, the cuts provide that only the dipole term for the pair with the small resolution parameter contributes. Note that the cancellation needed as any two of the partons $\{\hat{p}\}_{m+1}$ become collinear or one of them becomes soft is left intact.

We mention here a subtle point with respect to infrared singularities in eq. (6.3). Consider the $\{ij,k\}$ subtraction term. As we integrate over $\{\hat{p},\hat{f}\}_{m+1}$, we can encounter a point at which two of the momenta in $\{p\}_m^{ij,k}$ are collinear or one is zero. (The simplest situation is that two of the original $\{\hat{p}\}_{m+1}$ other than $\hat{p}_i$ and $\hat{p}_j$ are collinear or one is zero.) At this point, $\mathcal{D}_{ij,k} \propto \langle \mathcal{M}(\{p,f\}_m^{ij,k}) | \mathcal{M}(\{p,f\}_m^{ij,k})\rangle$ is singular. With an N-jet observable and with $m = N$, the function $F(\{p\}_m^{ij,k})$ vanishes at this point, so the singularity is weakened or eliminated. This is what happens in the standard dipole subtraction method [12]. However, suppose that $m > N$. Then $F(\{p\}_m^{ij,k})$ does not vanish at the point in question, but still $d_m(\{p,f\}_m^{ij,k})$ vanishes, so that the cut $d_{\rm ini} < d_m(\{p,f\}_m^{ij,k})$ eliminates the singularity.

In the virtual contribution, we write

$$\begin{aligned}\sigma_m^{\rm V+A} = \frac{1}{m!}\sum_{\{f\}_m}\int d\Gamma(\{p\}_m)\, \theta\big(d_{\rm ini} < d_m(\{p,f\}_m)\big) F_m(\{p\}_m) \times \\ \times\left\{V(\{p,f\}_m) - \sum_l \sum_{k\neq l} C_{l,k}(\{p,f\}_m, d_{\rm ini})\right\}.\end{aligned} \tag{6.6}$$

This follows eq. (4.7), with the addition of a cut $d_{\rm ini} < d_m$. There are also added terms involving functions $C_{l,k}$,

$$C_{l,k}(\{p,f\}_m, d_{\rm ini}) = \int_0^1 \frac{dy}{y}\int_0^1 dz \int_0^{2\pi}\frac{d\phi}{2\pi}\frac{1}{2}\sum_{\hat{f}_1,\hat{f}_2}\delta^f_{\hat{f}_1+\hat{f}_2}\,\theta\big(d_{\rm ini} < \tilde{d}(\{p,f\}_m, l, y, z)\big) \times$$

$$\times \frac{y(1-y)\,2p_l\cdot p_k}{16\pi^2}\,\mathcal{D}_{l,k}(\{p,f\}_m;Y)\,. \tag{6.7}$$

This arises because we modified the dipole subtractions in eq. (6.3) by imposing an extra cut $\tilde{d}(\{p,f\}_m^{ij,k},\{l,y,z\}_{ij,k}) < d_{\rm ini}$ that was not present in eq. (4.3). What we removed from the real emission subtraction we must also remove from the virtual loop subtraction. Doing that and changing integration variables from $\{\hat{p},\hat{f}\}_{m+1}$ to $(\{p,f\}_m;Y_l)$ gives eq. (6.6). The details of the variable transformation can be found in section 10, eq. (10.17). The functions $C_{l,k}(\{p,f\}_m,d_{\rm ini})$ are evaluated in section 12, eqs. (12.8), (12.9)

Using dipole subtraction with cuts as defined above, one could construct a computer code that would calculate at next-to-leading order the expectation values of infrared safe N-jet measurement functions for $N = 2,3,4,\dots$ up to the value for which one had the required calculated matrix elements. The program would produce partonic events with weights, and the same events could be used for the calculation of different N-jet observables with different values of N. The practical value of such a program would be minimal, since it would not add anything to having separate programs for each value of N. However, if one could add parton showers to the calculation thus organized, the program would have some added value over the separate perturbative programs. It is to this goal that we now turn.

7. Partial cross sections with showers

In the following sections, we discuss the construction of a parton shower algorithm that matches the dipole subtraction scheme for NLO calculations. We incorporate the k_T-jet matching scheme [11], so that the cross section is generated as a sum of partial cross sections σ_m as in sections 5 and 6, but this time with showers added. We suppose that we can calculate the perturbative m-parton cross section, $\sigma_m^{\rm B}$, at the Born level, $\alpha_{\rm s}^{B_m} \equiv \alpha_{\rm s}^{m-2}$ for $m = 2,3,\dots,m_{\rm max}$. Furthermore, we suppose that we can calculate the order $\alpha_{\rm s}^{B_m+1}$ corrections, $\sigma_m^{\rm R-A}$ and $\sigma_m^{\rm V+A}$, for $m = 2,3,\dots,m_{\rm NLO}$. The values of $m_{\rm NLO}$ and $m_{\rm max}$ depend on our ability to calculate, but we can always assume that $m_{\rm max} \ge m_{\rm NLO}+1$.

The cross section computed with parton showers will consist of contributions from each available m,

$$\sigma^{\rm NLO+S} = \sum_{m=2}^{m_{\rm NLO}} \left[\sigma_m^{\rm B+S} + \sigma_m^{\rm R+S} + \sigma_m^{\rm V+S}\right] + \sum_{m=m_{\rm NLO}+1}^{m_{\rm max}} \sigma_m^{\rm B+S}\,. \tag{7.1}$$

For the contributions at NLO level, there are three terms, which correspond to Born, real emission, and virtual loop terms with showers added (“$+S$”). For the remaining contributions there is only a Born term. We will arrange that (for a suitably behaved observable)

$$\begin{aligned}\sigma_m^{\rm B+S} + \sigma_m^{\rm R+S} + \sigma_m^{\rm V+S} = {}& \sigma_m^{\rm B} + \sigma_m^{\rm R-A} + \sigma_m^{\rm V+A} + \mathcal{O}(\alpha_{\rm s}^{B_m+2}) + \\ & + \mathcal{O}(\alpha_{\rm s}^{B_m} \times 1\,{\rm GeV}/\sqrt{s}\,)\end{aligned} \tag{7.2}$$

and

$$\sigma_m^{\rm B+S} = \sigma_m^{\rm B} + \mathcal{O}(\alpha_{\rm s}^{B_m+1}) + \mathcal{O}(\alpha_{\rm s}^{B_m} \times 1\,{\rm GeV}/\sqrt{s}\,)\,. \tag{7.3}$$

The main body of this construction is in the following section (with several subsections). There we construct $\sigma_m^{\rm B+S}$, $\sigma_m^{\rm R+S}$, and $\sigma_m^{\rm V+S}$ for $2 \le m \le m_{\rm NLO}$ and show that these quantities sum to $\sigma_m^{\rm B} + \sigma_m^{\rm R-A} + \sigma_m^{\rm V+A}$ to NLO accuracy. We also obtain the leading-order result in eq. (7.3) as a byproduct.

8. The shower construction

In this section, we discuss $\sigma_m^{\rm B+S}$, $\sigma_m^{\rm R+S}$, and $\sigma_m^{\rm V+S}$ for $2 \le m \le m_{\rm NLO}$ and show that these quantities sum to $\sigma_m^{\rm B} + \sigma_m^{\rm R-A} + \sigma_m^{\rm V+A}$ to NLO accuracy. We begin with $\sigma_m^{\rm B+S}$ in section 8.1 and the following subsections. Then $\sigma_m^{\rm R+S}$ is described in section 8.7. The virtual contribution $\sigma_m^{\rm V+A}$ is described in section 8.8. We sum the contributions in section 8.9.

8.1 Born term with showers

Our discussion begins in this subsection with $\sigma_m^{\rm B+S}$. We define

$$
\begin{aligned}
\sigma_m^{\rm B+S} = {} & \frac{1}{m!}\sum_{\{f\}_m}\int d\Gamma(\{p\}_m)\,\theta(d_{\rm ini} < d_m(\{p,f\}_m))\,W_m(\{p,f\}_m)\times \\
& \times \sum_{l=1}^{m}\sum_{k\neq l}\big\langle\mathcal{M}(\{p,f\}_m)\big|\int dY_l\,\boldsymbol{E}_{l,k}(\{p,f\}_m,Y_l)\,\big|\mathcal{M}(\{p,f\}_m)\big\rangle\times \\
& \times I(\{p,f\}_m;l,k,Y_l)\,.
\end{aligned}
\tag{8.1}
$$

This formula is illustrated in figure 1. The first line contains integrals over Born level parton momenta and a corresponding sum over parton flavors. The second line contains sums over choices l of the parton that splits and a spectator parton k along with an integral over the splitting variables Y_l and a matrix element of certain operators $\boldsymbol{E}_{l,k}$ acting on the Born amplitude $|\mathcal{M}\rangle$. The operators $\boldsymbol{E}_{l,k}$ together with the other factors in the formula describe the formation of showers from the Born level partons.

Our goal in the following subsections is first to explain the parts of eq. (8.1) and then to show that, with suitable definitions of the functions in eq. (8.1), the perturbative expansion of $\sigma_m^{\rm B+S}$ gives the Born cross section $\sigma_m^{\rm B}$ at order $\alpha_{\rm s}^{B_m}$. Expanding to one more order, we get certain order $\alpha_{\rm s}^{B_m+1}$ terms that we need to keep track of. We will then be able to add correction terms that allow us to match the $\alpha_{\rm s}^{B_m+1}$ contributions in the dipole subtraction scheme.

8.2 The Born level partons

The first line of eq. (8.1) concerns what we call the Born level partons. These are labelled by an index l that takes the values $l \in \{1,2,\dots,m\}$. The Born level partons are produced in the hard scattering and evolve into final state showers.

There is an integration over the phase space for the m final state partons with momenta $\{p\}_m$. There is also a sum over the flavors $\{f\}_m$ of the final state partons. The integration over the momenta $\{p\}_m$ is restricted by a factor

$$
\theta\big(d_{\rm ini} < d_m(\{p,f\}_m)\big)\,.
$$

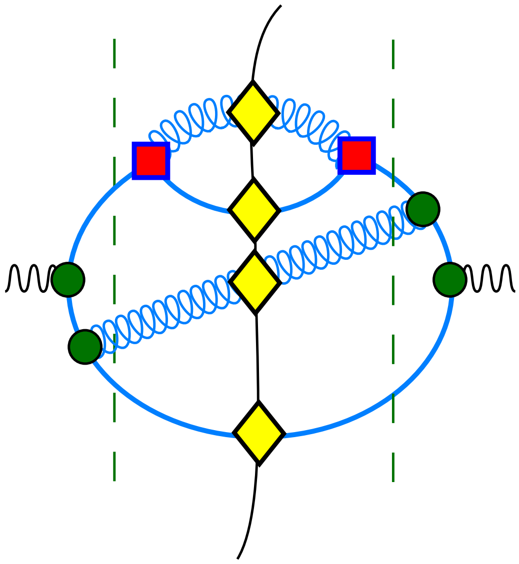

Figure 1: Illustration of $\sigma_3^{\mathrm{B+S}}$, eq. (8.1). The basic Born graph lies outside the dashed lines. The three partons are required to be resolvable at scale d_{ini}. To the left of the first dashed line, we illustrate one contribution to $|\mathcal{M}\rangle$. To the right, we illustrate one contribution to $\langle\mathcal{M}|$. We multiply by a reweighting factor W, not illustrated, that contains Sudakov factors for a constructed parton splitting history. Between the dashed lines there is a splitting of one of the partons l, using a spectator parton k, as described by the splitting operator $\boldsymbol{E}_{l,k}$. Finally, there is secondary showering represented in the figure by diamonds and in eq. (8.1) by the function I. The diamonds split the partons into a shower of partons and then into jets of hadrons.

Here $d_m(\{p,f\}_m)$ is defined by applying the k_T jet finding algorithm to the m parton momenta, as in eq. (5.3). As discussed in sections 5 and 6, we classify a final state of n partons by the values of the jet resolution parameters d_m. The contribution to the observable from states with $d_{m+1} < d_{\mathrm{ini}} < d_m$ are calculated using $\sigma_m^{\mathrm{B+S}}$ and its corrections. The cut on d_m in eq. (8.1) implements the first part of this general plan.

There is also a factor $W_m(\{p,f\}_m)$, which is the product of factors associated with the splitting history that matches the found jet structure, following the method of ref. [11]. Imagine that the found jet structure had been generated by parton shower Monte Carlo program. Then for each vertex V in the graph that represents the splitting history, there would be a factor $\alpha_{\mathrm{s}}(\sqrt{d_V s})$, where d_V is the k_T scale of the splitting at that vertex. In the exact squared matrix element $\langle\mathcal{M}(\{p,f\}_m)|\mathcal{M}(\{p,f\}_m)\rangle$, there is a factor $\alpha_{\mathrm{s}}(\mu_{\mathrm{R}})$ for each vertex. We switch from $\alpha_{\mathrm{s}}(\mu_{\mathrm{R}})$ to $\alpha_{\mathrm{s}}(\sqrt{d_V s})$ by including in $W_m(\{p,f\}_m)$ a factor

$$\frac{\alpha_{\mathrm{s}}(\sqrt{d_V s})}{\alpha_{\mathrm{s}}(\mu_{\mathrm{R}})} \tag{8.2}$$

for each vertex. The parton shower Monte Carlo program would also generate a Sudakov factor that represents the probability that the partons did not split between each scale d_V at which a splitting occurred and the next smaller scale d_V and the probability that the partons did not split between the smallest scale d_V at which a splitting occurred and the limiting scale $d_{\rm ini}$. This Sudakov factor is also included in $W_m(\{p,f\}_m)$. The details are given in section 13.

All that we need to know at present about $W_m(\{p,f\}_m)$ is that it has a perturbative expansion in powers of $\alpha_{\rm s}(\mu_{\rm R})$ so that

$$W_m(\{p,f\}_m) = 1 + \frac{\alpha_{\rm s}(\mu_{\rm R})}{2\pi}\, W_m^{(1)}(\{p,f\}_m) + \cdots\,. \tag{8.3}$$

We will use this property in our perturbative analysis.

In the second line of eq. (8.1) there is a matrix element of certain operators $\boldsymbol{E}_{l,k}$ that act on the Born amplitude $\big|\mathcal{M}(\{p,f\}_m)\big\rangle$ to generate the splitting of the partons. Our next task is to describe this parton splitting.

8.3 Description of parton splitting

Each Born level parton has the opportunity to split into two daughter partons. The one that splits with the largest (suitably defined) evolution parameter is designated as parton l. The splitting of the remaining partons is left to be described by the function I in eq. (8.1). This splitting of parton l with the help of spectator parton k is described by an operator $\boldsymbol{E}_{l,k}$.

There are a number of parameters that describe the splitting of parton l,

$$Y_l = \{y_l, z_l, \phi_l, \hat f_{l,1}, \hat f_{l,2}\}\,. \tag{8.4}$$

as in eq. (3.9). We integrate over these variables in eq. (8.1), using $\int dY_l$ as defined in eq. (3.10).

The momenta and flavors of the $m+1$ daughter partons are given in terms of the momenta and flavors of the m mother partons and the splitting variables Y_l by the transformation (3.7),

$$\{\hat p, \hat f\}_{m+1} = \mathcal{R}_{l,k}(\{p,f\}_m, Y_l)\,. \tag{8.5}$$

8.4 The splitting functions

Now, we turn to the splitting function $\boldsymbol{E}_{l,k}$ in eq. (8.1), which is an operator on the color and spin space of parton l in the vector $|\mathcal{M}\rangle$. This operator has the following form,

$$\begin{aligned}
\boldsymbol{E}_{l,k} = {} & \int_0^\infty dr\, \delta(r - R_l(\{p,f\}_m, y_l, z_l))\, \theta(\tilde d(\{p,f\}_m, l, y_l, z_l) < d_{\rm ini}) \times \\
& \times \frac{\boldsymbol{T}_l\cdot\boldsymbol{T}_k}{-\boldsymbol{T}_l^2}\, \frac{\alpha_{\rm s}(k_T(\{p,f\}_m, l, y, z))}{2\pi}\, \boldsymbol{S}_l(p_l, f_l, Y_l) \times \\
& \times \exp\biggl(- \int_r^\infty dr' \int_0^1 \frac{dy'}{y'} \int_0^1 dz' \sum_{l'} \delta(r' - R_{l'}(\{p,f\}_m, y', z')) \times \\
& \qquad \times \theta(\tilde d(\{p,f\}_m, l', y', z') < d_{\rm ini})\, \frac{\alpha_{\rm s}(k_T(\{p,f\}_m, l', y', z'))}{2\pi} \times \\
& \qquad \times \big\langle \boldsymbol{S}(y', z', f_{l'}) \big\rangle \biggr)\,.
\end{aligned} \tag{8.6}$$

The parton splitting is organized according to an evolution parameter r, which is defined in eq. (8.6) to be a certain function R_l of the hard parton momenta and flavors $\{p, f\}_m$ and the splitting parameters y_l, z_l. The function R_l measures the hardness of the splitting. There is some flexibility in choosing this function as long as $y \to 0$ implies $R_l \to 0$. Some of the choices available may be understood using the kinematic variables described in section 10. If the secondary shower encoded in the Monte Carlo interface function to be described in section 8.5 is based on the PYTHIA algorithm it might be sensible to set R_l to a scaled virtuality,

$$R_l(\{p, f\}_m, y, z) = s_l\, y\,, \tag{8.7}$$

where s_l is the virtuality scale associated with parton l as discussed at eq. (6.4). In the HERWIG case one might choose a scaled squared transverse momentum,

$$R_l(\{p, f\}_m, y, z) = s_l\, yz(1-z)\,. \tag{8.8}$$

We have indicated the scale of α_s as a function k_T, defined by

$$k_T(\{p, f\}_m, l, y, z) = [s_l\, yz(1-z)]^{1/2}\,. \tag{8.9}$$

Other choices for the scale in α_s are possible. The matching to fixed order perturbation theory will work as long as we can write $\alpha_s(k_T(\{p, f\}_m, l, y, z)) = \alpha_s(\mu_R) + \mathcal{O}(\alpha_s^2)$. Although our notation allows the possibility of a variety of choices for the functions R_l and k_T, for purposed of minimizing the dependence on the arbitrary parameter $d_{\rm ini}$ our favored choices are given by eqs. (8.8) and (8.9) with s_l a certain function of $\{p, f\}_m$ that is defined later in eq. (13.16).

In eq. (8.6), we also make use of a function $\tilde{d}$ defined in eq. (6.4) that gives an approximate version of the resolution variable associated with the splitting generated with the splitting parameters y_l, z_l. With the use of this function, we limit the splitting in $\boldsymbol{E}_l$ to be unresolvable at a scale d that is approximately $d_{\rm ini} \times 2p_l \cdot p_{k_l}/s_l$ (see eq. (6.5)).

In each $\boldsymbol{E}_{l,k}$ operator, there is an operator on the parton color space, $\boldsymbol{T}_l \cdot \boldsymbol{T}_k/[-\boldsymbol{T}_l^2]$, that is familiar from the dipole splitting formulas. Next, there is an operator on the parton spin space, $\boldsymbol{S}_l$, with the interpretation that $(\alpha_s/(2\pi))\,\boldsymbol{S}_l$ is the probability for the parton to split at a given evolution parameter r if it has not split at a higher evolution parameter. The splitting function $\boldsymbol{S}_l$ depends on the splitting parameters $Y_l = \{y_l, z_l, \phi_l, \hat{f}_{l,1}, \hat{f}_{l,2}\}$ for parton l as well as on the momentum p_l, which is needed to fully specify the meaning of ϕ_l. We will specify this function later in section 11, eq. (11.4).

The next factor, the Sudakov exponential, gives the probability that *none* of the partons has split at a higher evolution scale. Thus we work in a scheme similar to that of Sjöstrand and Skands [14] and of Nason [15], picking out the hardest splitting. In the exponent there is a sum over partons l' and an integration over virtualities y' and the momentum fractions z' of the "virtual" splittings. The corresponding evolution parameter r' is required to be bigger than r.

The remaining factor in the Sudakov exponent is the average over angle and flavors of $\boldsymbol{S}$ for parton l',

$$\big\langle \boldsymbol{S}(y_{l'}, z_{l'}, f_{l'})\big\rangle \equiv \int \frac{d\phi_l}{2\pi}\,\frac{1}{2}\sum_{\hat f_{l',1},\hat f_{l',2}} \delta^{f_{l'}}_{\hat f_{l',1}+\hat f_{l',2}}\,\boldsymbol{S}_{l'}(p_{l'}, y_{l'}, z_{l'}, \phi_{l'}, \hat f_{l',1}, \hat f_{l',2})\,. \tag{8.10}$$

With our definitions, this is a numerical function times a unit operator on the partonic spin space. Explicit expressions are given later in section 11, eq. (11.11), (11.12).

8.5 Monte Carlo interface function

The last factor in eq. (8.1) is

$$I(\{p, f\}_m; l, k, Y_l)\,. \tag{8.11}$$

We imagine that after the parton splitting represented by the splitting function $\boldsymbol{E}_{l,k}$, which we may call the *primary* splitting, there is further parton showering, which we may call *secondary showering*. This showering is to be carried out by a shower Monte Carlo style computer program. The factor I represents the average value of the observable corresponding to the daughter hadrons after secondary showering when the shower starts with initial conditions specified by the variables $(\{p, f\}_m; l, k, Y_l)$. Here the initial conditions include not only the partonic state

$$\{\hat p, \hat f\}_{m+1} = \mathcal{R}_{l,k}(\{p, f\}_m, Y_l) \tag{8.12}$$

generated from the partonic state $\{p, f\}_m$ by splitting parton l according to the splitting variables Y_l, but also the history of the first step of showering as specified by Y_l.

We can think of I as being an integral,

$$\begin{aligned} I = \sum_N \frac{1}{N!} \sum_{\{\tilde f\}_N} \prod_{i=1}^{N} \bigg(\int d\tilde p_i\ \delta(\tilde p_i^2 - m^2(\tilde f_i))\bigg) \times \\ \times P(\{\tilde p, \tilde f\}_N | \{p, f\}_m; l, k, Y_l)\, F_N(\{\tilde p\}_N)\,. \end{aligned} \tag{8.13}$$

There is a sum over the number N of final state hadrons that are generated by the shower, a symmetry factor $1/N!$, a sum over the flavors $\{\tilde f\}_N$ of these hadrons and an integration over their momenta $\{\tilde p\}_N$. In the next line there is a factor P that represents the probability density to produce the final state hadrons given the starting conditions represented by $(\{p, f\}_m; l, k, Y_l)$. The final factor in eq. (8.13) is the measurement function evaluated with the produced final state hadrons. If one were to substitute $F = 1$, one would get the normalization condition for the conditional probability P, namely $I = 1$. Equivalently, if we substitute $F = \lambda$, then

$$I(\{p, f\}_m; l, k, Y_l)\big|_{F=\lambda} = \lambda\,. \tag{8.14}$$

Here λ could depend on the initial conditions for the secondary shower as represented by the variables $(\{p, f\}_m; l, k, Y_l)$.

A simple model for I is obtained by omitting all secondary showering. Then I becomes

$$I^{(0)}(\{p, f\}_m; l, k, Y_l) = F_{m+1}(\mathcal{R}_{l,k}(\{p, f\}_m, Y_l))\,. \tag{8.15}$$

JHEP10(2005)024

We will have more to say about the construction of the Monte Carlo showering program represented by function I in section 15. One important feature is that the primary splitting, the splitting of parton l, should be the hardest splitting in the shower (according to the hardness measure R in eq. (8.6)). However, we leave the choice of this program largely open. Here we simply note two properties that I should have when the measurement F is an infrared safe N-jet measurement function.

First, secondary showering should provide perturbative and power suppressed corrections to the simple function $I^{(0)}$ (assuming, always, an infrared safe observable):

$$\begin{aligned} I(\{p,f\}_m; l,k,Y_l) = {} & F_{m+1}(\mathcal{R}_{l,k}(\{p,f\}_m, Y_l)) \times \\ & \times [1 + \mathcal{O}(\alpha_s) + \mathcal{O}(1\,\mathrm{GeV}/\sqrt{s})]\,. \end{aligned} \tag{8.16}$$

Here the order α_s correction corresponds to splitting with a substantial virtuality, while the power suppressed correction corresponds to hadronization.

Second, when y_l vanishes, I should reduce to $I^{(0)}$ with only power corrections,

$$I(\{p,f\}_m; l,k,0,z_l,\phi_l,\hat{f}_{l,1},\hat{f}_{l,2}) = F_m(\{p\}_m) \times [1 + \mathcal{O}(1\,\mathrm{GeV}/\sqrt{s})]\,. \tag{8.17}$$

The requirement here is that having the initial splitting virtuality y_l equal to zero should set the maximum hardness for all of the secondary splittings to zero and thus turn the secondary showering off except for hadronization. The hadronization model should then turn the partons into jets of a limited mass, leading to only power suppressed contributions to the measurement function. Note that $F_{m+1}(\{\hat{p},\hat{f}\}_{m+1})$ reduces to $F_m(\{p\}_m)$ here because of the infrared safety property of the measurement function.

In sections 8.7 and 8.8, we will also need a function $\tilde{I}(\{p,f\}_n)$ that represents the average value of the observable corresponding to the daughter hadrons after secondary showering when the shower starts with a partonic state $\{p,f\}_n$ with no other information (other than the d_{ini} cut) given as to previous shower history. The function $\tilde{I}(\{p,f\}_n)$ obeys the normalization condition

$$\tilde{I}(\{p,f\}_n)\big|_{F=\lambda} = \lambda\,, \tag{8.18}$$

where λ could depend on the initial conditions for the secondary shower as represented by the variables $(\{p,f\}_n)$.

As in the case of the version of I above, we assume that $\tilde{I}$ is constructed so that the effects on the measurement of the showering are suppressed either by a power of α_s or a power of $1\,\mathrm{GeV}/\sqrt{s}$,

$$\tilde{I}(\{p,f\}_n) = F_n(\{p\}_n) \times [1 + \mathcal{O}(\alpha_s) + \mathcal{O}(1\,\mathrm{GeV}/\sqrt{s})]\,. \tag{8.19}$$

8.6 Perturbative expansion

We now seek the perturbative expansion of $\sigma_m^{\mathrm{B+S}}$ in eq. (8.1). Insert into eq. (8.1) a factor $1 = T_l + \Delta_l$, where T_l sets y_l to zero in I and $\Delta_l \equiv 1 - T_l$. Then

$$\sigma_m^{\mathrm{B+S}} = \frac{1}{m!} \sum_{\{f\}_m} \int d\Gamma(\{p\}_m)\,\theta\big(d_{\mathrm{ini}} < d_m(\{p,f\}_m)\big)\, W_m(\{p,f\}_m) \times$$

$$\times \sum_{l=1}^{m} \sum_{k\neq l} \big\langle \mathcal{M}(\{p,f\}_m) \big| \int dY_l\, \boldsymbol{E}_{l,k}(\{p,f\}_m, Y_l) \big| \mathcal{M}(\{p,f\}_m) \big\rangle \times$$
$$\times (T_l + \Delta_l) I(\{p,f\}_m; l,k,Y_l)\,. \tag{8.20}$$

Our strategy will be to separate the T_l term from the Δ_l term. We call the the T_l term $\sigma_{m,T}^{\rm B+S}$ and the Δ_l term $\sigma_{m,\Delta}^{\rm B+S}$. Thus we write

$$\sigma_m^{\rm B+S} = \sigma_{m,T}^{\rm B+S} + \sigma_{m,\Delta}^{\rm B+S}\,. \tag{8.21}$$

Then we will expand each term in powers of $\alpha_{\rm s}$, up to next-to-leading order. We will find that, to this order, there are two terms in the expansion of $\sigma_{m,T}^{\rm B+S}$

$$\sigma_{m,T}^{\rm B+S} = \sigma_{m,\{T,0\}}^{\rm B+S} + \sigma_{m,\{T,1\}}^{\rm B+S} + \mathcal{O}(\alpha_{\rm s}^{B_m+2})\,. \tag{8.22}$$

Here $\sigma_{m,\{T,0\}}^{\rm B+S}$ is proportional to $\alpha_{\rm s}^{B_m}$ and is, in fact, the Born contribution, $\sigma_m^{\rm B}$, to σ_m. Then $\sigma_{m,\{T,1\}}^{\rm B+S}$ consists of certain $\alpha_{\rm s}^{B_m+1}$ corrections. We will then find that the expansion of $\sigma_{m,\Delta}^{\rm B+S}$ begins at order $\alpha_{\rm s}^{B_m+1}$, so that we need only the first term in this expansion in order to evaluate $\sigma_m^{\rm B+S}$ to order $\alpha_{\rm s}^{B_m+1}$.

8.6.1 The T_l contribution

We begin by analyzing the term $\sigma_{m,T}^{\rm B+S}$. We use

$$T_l\, I(\{p,f\}_m, l,k,Y_l)) \equiv I(\{p,f\}_m; l,k,0,z_l,\phi_l,\hat f_{l,1},\hat f_{l,2})\,. \tag{8.23}$$

Using the property (8.17) of I we have

$$I(\{p,f\}_m; l,k,0,z_l,\phi_l,\hat f_{l,1},\hat f_{l,2}) = F\big(\{p\}_m\big) \times [1 + \mathcal{O}\left(1\,{\rm GeV}/\sqrt{s}\right)]\,. \tag{8.24}$$

Then

$$\sigma_{m,T}^{\rm B+S} = \frac{1}{m!} \sum_{\{f\}_m} \int d\mathit{\Gamma}(\{p\}_m) \theta\big(d_{\rm ini} < d_m(\{p,f\}_m)\big) W_m(\{p,f\}_m)\, F_m\big(\{p\}_m\big) \times$$
$$\times \sum_{l=1}^{m} \sum_{k\neq l} \big\langle \mathcal{M}(\{p,f\}_m) \big| \int dY_l\, \boldsymbol{E}_{l,k}(\{p,f\}_m, Y_l) \big| \mathcal{M}(\{p,f\}_m) \big\rangle \times$$
$$\times [1 + \mathcal{O}(1\,{\rm GeV}/\sqrt{s})]\,. \tag{8.25}$$

Let us look at the sum of the $\boldsymbol{E}_{l,k}$ operators integrated over the corresponding splitting variables. We have

$$\sum_{l,k} \int dY_l\, \boldsymbol{E}_{l,k} = \sum_{l=1}^{m} \sum_{k\neq l} \frac{\boldsymbol{T}_l \cdot \boldsymbol{T}_k}{-\boldsymbol{T}_l^2} \int_0^\infty dr \int_0^1 \frac{dy_l}{y_l} \int_0^1 dz_l\, \delta(r - R_l(\{p,f\}_m, y_l, z_l)) \times$$
$$\times \theta(\tilde d(\{p,f\}_m, l, y_l, z_l) < d_{\rm ini}) \frac{\alpha_{\rm s}(k_T(\{p,f\}_m, l, y, z))}{2\pi}$$
$$\times \frac{1}{2} \sum_{\hat f_{l,1}, \hat f_{l,2}} \delta^{f_l}_{\hat f_{l,1} + \hat f_{l,2}} \int_0^{2\pi} \frac{d\phi_l}{2\pi}\, \boldsymbol{S}_l(p_l, f_l, Y_l) \times$$

$$
\begin{aligned}
&\times \exp\biggl(-\int_r^\infty dr' \int_0^1 \frac{dy'}{y'} \int_0^1 dz' \sum_{l'} \delta(r' - R_l(\{p,f\}_m, y', z')) \times \\
&\quad \times \theta(\tilde{d}(\{p,f\}_m, l, y', z') < d_{\rm ini})\ \frac{\alpha_{\rm s}(k_T(\{p,f\}_m, l', y', z'))}{2\pi} \times \\
&\quad \times \bigl\langle \boldsymbol{S}(y', z', f_{l'}) \bigr\rangle \biggr)\,,
\end{aligned} \tag{8.26}
$$

where $\langle \boldsymbol{S} \rangle$ was defined in eq. (8.10). According to eq. (4.6), the sum over k of the color factors is

$$
\sum_k \frac{\boldsymbol{T}_l \cdot \boldsymbol{T}_k}{-\boldsymbol{T}_l^2} = 1 \tag{8.27}
$$

when operating on the state $\bigl|\mathcal{M}(\{p,f\}_m)\bigr\rangle$. Having eliminated the color factor, we have an integral of a derivative,

$$
\begin{aligned}
&\int_0^\infty dr\, \frac{d}{dr} \exp\biggl(-\int_r^\infty dr' \int_0^1 \frac{dy'}{y'} \int_0^1 dz'\, \delta(r' - R_{l'}(\{p,f\}_m, y', z')) \times \\
&\times \theta(\tilde{d}(\{p,f\}_m, l', y', z') < d_{\rm ini}) \frac{\alpha_{\rm s}(k_T(\{p,f\}_m, l', y', z'))}{2\pi} \bigl\langle \boldsymbol{S}(y', z', f_{l'}) \bigr\rangle \biggr) = 1\,,
\end{aligned} \tag{8.28}
$$

since the exponent is 0 at $r = \infty$ and $-\infty$ at $r = 0$ (since the integration region for $r = 0$ includes the $y \to 0$ singularity). Thus (when operating on the state $\bigl|\mathcal{M}(\{p,f\}_m)\bigr\rangle$)

$$
\sum_{l,k} \int dY_l\, \boldsymbol{E}_{l,k} = 1\,. \tag{8.29}
$$

We consider next the factor $W_m(\{p,f\}_m)$ in eq. (8.25), which we write as 1 plus an order $\alpha_{\rm s}$ contribution $\alpha_{\rm s}\, W_m^{(1)}(\{p,f\}_m)$ according to eq. (8.3),

$$
W_m(\{p,f\}_m) = 1 + \frac{\alpha_{\rm s}(\mu_{\rm R})}{2\pi}\, W_m^{(1)}(\{p,f\}_m) + \cdots\,. \tag{8.30}
$$

Taking the lowest order term in $\sigma_{m,T}^{\rm B+S}$ gives the lowest order term in the perturbative expansion of $\sigma_m^{\rm B+S}$. This term is

$$
\begin{aligned}
\sigma_{m,\{T,0\}}^{\rm B+S} = \frac{1}{m!} \sum_{\{f\}_m} \int & d\Gamma(\{p\}_m)\ \theta\bigl(d_{\rm ini} < d_m(\{p,f\}_m)\bigr) \times \\
&\times \bigl\langle \mathcal{M}(\{p,f\}_m) \bigl| \mathcal{M}(\{p,f\}_m) \bigr\rangle\, F_m\bigl(\{p\}_m\bigr) \times \\
&\times \bigl[1 + \mathcal{O}(1\,{\rm GeV}/\sqrt{s})\bigr]\,.
\end{aligned} \tag{8.31}
$$

Comparing to eq. (6.2), we see that

$$
\sigma_{m,\{T,0\}}^{\rm B+S} = \sigma_m^{\rm B} \times \bigl[1 + \mathcal{O}(1\,{\rm GeV}/\sqrt{s})\bigr]\,. \tag{8.32}
$$

The only order $\alpha_{\rm s}^1$ term in $\sigma_{m,T}^{\rm B+S}$ comes from the expansion of the reweighting function W. This term is

$$
\sigma_{m,\{T,1\}}^{\rm B+S} = \frac{1}{m!} \sum_{\{f\}_m} \int d\Gamma(\{p\}_m)\, \theta\bigl(d_{\rm ini} < d_m(\{p,f\}_m)\bigr)\, F_m(\{p\}_m) \times
$$

$$\times \frac{\alpha_s(\mu_R)}{2\pi}\, W_m^{(1)}(\{p,f\}_m)\, \big\langle \mathcal{M}(\{p,f\}_m)\big|\mathcal{M}(\{p,f\}_m)\big\rangle \times$$
$$\times \big[1 + \mathcal{O}(1\,\mathrm{GeV}/\sqrt{s})\big]\,, \tag{8.33}$$

where $W_m^{(1)}$ is the first order contribution to W, eq. (8.3).

8.6.2 The Δ_l contribution

Now we turn to the contribution $\sigma_{m,\Delta}^{\mathrm{B+S}}$ in eq. (8.21). From the definition in eq. (8.21) we have

$$\begin{aligned}\sigma_{m,\Delta}^{\mathrm{B+S}} = {} & \frac{1}{m!}\sum_{\{f\}_m}\int d\Gamma(\{p\}_m)\,\theta\big(d_{\mathrm{ini}} < d_m(\{p,f\}_m)\big)\, W_m(\{p,f\}_m) \times \\ & \times \sum_{l=1}^{m}\sum_{k\neq l}\big\langle \mathcal{M}(\{p,f\}_m)\big| \int dY_l\, \boldsymbol{E}_{l,k}(\{p,f\}_m, Y_l)\,\big|\mathcal{M}(\{p,f\}_m)\big\rangle \times \\ & \times \times\, \Delta_l I(\{p,f\}_m; l,k,Y_l)\,. \end{aligned}\tag{8.34}$$

Here

$$\Delta_l\, I(\{p,f\}_m; l,k,Y_l) = I(\{p,f\}_m; l,k,Y_l) - I(\{p,f\}_m; l,k,0,z_l,\phi_l,\hat{f}_{l,1},\hat{f}_{l,2})\,. \tag{8.35}$$

For I with $y_l = 0$ we can use eq. (8.24). For I with nonzero y_l we use eq. (8.16). This gives

$$\begin{aligned}\Delta_l\, I(\{p,f\}_m; l,k,Y_l) = {} & \Big\{F_{m+1}(\mathcal{R}_{l,k}(\{p,f\}_m, Y_l)) - F_m\big(\{p\}_m\big)\Big\} \times \\ & \times[1 + \mathcal{O}(\alpha_s) + \mathcal{O}\left(1\,\mathrm{GeV}/\sqrt{s}\right)]\,. \end{aligned}\tag{8.36}$$

We use this result in eq. (8.34), giving

$$\begin{aligned}\sigma_{m,\Delta}^{\mathrm{B+S}} = {} & \frac{1}{m!}\sum_{\{f\}_m}\int d\Gamma(\{p\}_m)\,\theta\big(d_{\mathrm{ini}} < d_m(\{p,f\}_m)\big)\, W_m(\{p,f\}_m) \times \\ & \times \sum_{l=1}^{m}\sum_{k\neq l}\big\langle \mathcal{M}(\{p,f\}_m)\big| \int dY_l\, \boldsymbol{E}_{l,k}(\{p,f\}_m, Y_l)\,\big|\mathcal{M}(\{p,f\}_m)\big\rangle \times \\ & \times\Big\{F_{m+1}(\mathcal{R}_{l,k}(\{p,f\}_m, Y_l)) - F_m\big(\{p\}_m\big)\Big\} \times \\ & \times[1 + \mathcal{O}(\alpha_s) + \mathcal{O}\left(1\,\mathrm{GeV}/\sqrt{s}\right)]\,. \end{aligned}\tag{8.37}$$

As we will see, the leading contribution to $\sigma_{m,\Delta}^{\mathrm{B+S}}$ is of order $\alpha_s^{B_m+1}$. Therefore, we can expand all of the factors and keep only the leading order terms. Using eq. (8.3) we can replace W by 1. We then use eq. (8.6) for $\boldsymbol{E}_{l,k}$, replacing the Sudakov exponential by 1, since we want only the first perturbative contribution (and since, because of the subtraction at $y_l = 0$, the integrand is not divergent at $y_l = 0$). We can also replace the running coupling $\alpha_s(k_T)$ in $\boldsymbol{E}_{l,k}$ by $\alpha_s(\mu_R)$ at leading perturbative order. Then

$$\sigma_{m,\Delta}^{\mathrm{B+S}} = \frac{1}{m!}\sum_{\{f\}_m}\int d\Gamma(\{p\}_m)\;\theta\big(d_{\mathrm{ini}} < d_m(\{p,f\}_m)\big) \times$$

$$\times \sum_{l=1}^{m} \sum_{k \neq l} \int dY_l\, \theta(\tilde{d}(\{p,f\}_m, l, y_l, z_l) < d_{\rm ini}) \times$$
$$\times \frac{\alpha_{\rm s}(\mu_{\rm R})}{2\pi} \langle \mathcal{M}(\{p,f\}_m) \big| \frac{\boldsymbol{T}_l \cdot \boldsymbol{T}_k}{-\boldsymbol{T}_l^2} \boldsymbol{S}_l(p_l, f_l, Y_l) \big| \mathcal{M}(\{p,f\}_m) \rangle \times$$
$$\times \Big\{ F_{m+1}(\mathcal{R}_{l,k}(\{p,f\}_m, Y_l)) - F_m\big(\{p\}_m\big) \Big\} \times$$
$$\times [1 + \mathcal{O}(\alpha_{\rm s}) + \mathcal{O}\left(1\,{\rm GeV}/\sqrt{s}\right)]\,. \tag{8.38}$$

In this equation there is an integration $\int dY_l$, which we can write out in full using eq. (3.10). Then we can change integration variables to momenta $\{\hat{p}\}_{m+1}$ after the splitting described by Y_l. The jacobian is given in section 10, eq. (10.17). After the change of variables, we have

$$\sigma^{\rm B+S}_{m,\Delta} = \frac{1}{(m+1)!} \sum_{\{f\}_{m+1}} \int d\Gamma(\{\hat{p}\}_{m+1}) \times$$
$$\times \sum_{\substack{i,j\\ \text{pairs}}} \sum_{k \neq i,j} \theta(\tilde{d}(\{p,f\}^{ij,k}_m, \{l,y,z\}_{ij,k}) < d_{\rm ini} < d_m(\{p,f\}^{ij,k}_m)) \times$$
$$\times \frac{\alpha_{\rm s}(\mu_{\rm R})}{2\pi} \frac{16\pi^2}{2\hat{p}_i \cdot \hat{p}_j\,(1 - y_{ij,k})} \times$$
$$\times \langle \mathcal{M}(\{p,f\}^{ij,k}_m) \big| \frac{\boldsymbol{T}_{ij} \cdot \boldsymbol{T}_k}{-\boldsymbol{T}_{ij}^2} \boldsymbol{S}_{ij}(\{p_{ij}, f_{ij}, Y_{ij}\}_{ij,k}) \big| \mathcal{M}(\{p,f\}^{ij,k}_m) \rangle \times$$
$$\times \Big\{ F_{m+1}(\{\hat{p}\}_{m+1}) - F_m\big(\{p\}^{ij,k}_m\big) \Big\} \times$$
$$\times [1 + \mathcal{O}(\alpha_{\rm s}) + \mathcal{O}\left(1\,{\rm GeV}/\sqrt{s}\right)]\,. \tag{8.39}$$

Using eq. (4.5),

$$\mathcal{D}_{ij,k}\big(\{\hat{p},\hat{f}\}_{m+1}\big) =$$
$$= \frac{1}{2\hat{p}_i \cdot \hat{p}_j} \left\langle \mathcal{M}\big(\{p,f\}^{ij,k}_m\big) \right| \frac{\boldsymbol{T}_{ij} \cdot \boldsymbol{T}_k}{-\boldsymbol{T}_{ij}^2} \boldsymbol{V}_{ij}(\{\tilde{p}_{ij}, y, z, \phi\}_{ij,k}, f_i, f_j) \left| \mathcal{M}\big(\{p,f\}^{ij,k}_m\big) \right\rangle, \tag{8.40}$$

we see that if we identify

$$\frac{\alpha_{\rm s}}{2\pi} \frac{16\pi^2}{1-y} \boldsymbol{S}_l(p,f,Y) = \boldsymbol{V}_l(\{p,y,z,\phi\}, f_1, f_2)\,, \tag{8.41}$$

we will have

$$\sigma^{\rm B+S}_{m,\Delta} = \frac{1}{(m+1)!} \sum_{\{f\}_{m+1}} \int d\Gamma(\{\hat{p}\}_{m+1}) \sum_{\substack{i,j\\ \text{pairs}}} \sum_{k \neq i,j} \mathcal{D}_{ij,k}\big(\{\hat{p},\hat{f}\}_{m+1}\big) \times$$
$$\times \theta\big(\tilde{d}(\{p,f\}^{ij,k}_m, \{l,y,z\}_{ij,k}) < d_{\rm ini} < d_m(\{p,f\}^{ij,k}_m)\big) \times$$
$$\times \Big\{ F_{m+1}(\{\hat{p}\}_{m+1}) - F_m\big(\{p\}^{ij,k}_m\big) \Big\} \times$$
$$\times [1 + \mathcal{O}(\alpha_{\rm s}) + \mathcal{O}\left(1\,{\rm GeV}/\sqrt{s}\right)]\,. \tag{8.42}$$

We will use $\sigma^{\rm B+S}_{m,\Delta}$ in this form to combine with $\sigma^{\rm R+S}_m$, which is discussed in the following section.

8.7 NLO real emission corrections with shower

We turn to the discussion of the NLO corrections. Let us start with the real contribution. Define

$$
\begin{aligned}
\sigma_m^{\rm R+S} = \frac{1}{(m+1)!} \sum_{\{\hat f\}_{m+1}} \int d\Gamma(\{\hat p\}_{m+1})\, \tilde I\big(\{\hat p, \hat f\}_{m+1}\big) W_m^{\rm R+S}(\{p,f\}_{m+1}) \times \\
\times\Big\{ \big\langle \mathcal{M}(\{\hat p, \hat f\}_{m+1}) \big| \mathcal{M}(\{\hat p, \hat f\}_{m+1}) \big\rangle \times \\
\times\, \theta\big(d_{m+1}(\{\hat p, \hat f\}_{m+1}) < d_{\rm ini} < d_m(\{\hat p, \hat f\}_{m+1})\big) - \\
- \sum_{\substack{i,j \\ \text{pairs}}} \sum_{k\neq i,j} \mathcal{D}_{ij,k}\big(\{\hat p, \hat f\}_{m+1}\big) \times \\
\times\, \theta\big(\tilde d(\{p,f\}_m^{ij,k}, \{l,y,z\}_{ij,k}) < d_{\rm ini} < d_m(\{p,f\}_m^{ij,k})\big)\Big\}\,. \qquad (8.43)
\end{aligned}
$$

This formula is illustrated in figure 2. The first term is the $m+1$-parton matrix element squared and the second term is the sum of the dipole contributions to eliminate the infrared singularities. There is a Monte Carlo interface function $\tilde I$ with the property eq. (8.19). There is a reweighting factor $W_m^{\rm R+S}(\{p,f\}_{m+1})$ that is similar to the reweighting factor W in $\sigma_m^{\rm B+S}$. We discuss this factor in section 13, eq. (13.19). All that we need to know here is that $W_m^{\rm R+S}(\{p,f\}_{m+1})$ has a perturbative expansion that begins with 1: $W_m^{\rm R+S}(\{p,f\}_{m+1}) = 1 + \mathcal{O}(\alpha_{\rm s})$. There is a cut on d_m and d_{m+1}. The $m+1$-parton state should be not resolvable at scale $d_{\rm ini}$ but, having put the two closest partons together, the resulting m-parton state should be resolvable. In the dipole terms we have cuts on d_m and on $\tilde d_{m+1}$ for the splitting in question, eq. (6.4). The m-parton states defined by the dipole momentum set $\{p\}_m^{ij,k}$ should be resolvable at the scale $d_{\rm ini}$. However, every dipole splitting gives only non-resolvable radiations according to the $\tilde d$ measure.

In eq. (8.43) there is a potential infrared singularity when two of the partons, $\hat p_i$ and $\hat p_j$, become collinear or one vanishes. This singularity is rendered harmless by the corresponding $\{ij,k\}$ subtraction terms. Each of the $\{ij,k\}$ subtraction terms can have other singularities when two of the momenta $\{p\}_m^{ij,k}$ are collinear or one is zero. However at such an extra singularity, $d_m(\{p,f\}_m^{ij,k})$ vanishes, so that the cut $d_{\rm ini} < d_m(\{p,f\}_m^{ij,k})$ eliminates the singularity.

The perturbative expansion of this is simple. Using eq. (8.19), we can replace $\tilde I$ by the jet observable function F and, using $W_m^{\rm R+S}(\{p,f\}_{m+1}) = 1 + \mathcal{O}(\alpha_{\rm s})$, we can replace W by 1. This gives

$$
\begin{aligned}
\sigma_m^{\rm R+S} = \frac{1}{(m+1)!} \sum_{\{\hat f\}_{m+1}} \int d\Gamma(\{\hat p\}_{m+1})\, F_{m+1}\big(\{\hat p\}_{m+1}\big) \times \\
\times\Big\{ \big\langle \mathcal{M}(\{\hat p, \hat f\}_{m+1}) \big| \mathcal{M}(\{\hat p, \hat f\}_{m+1}) \big\rangle \times \\
\times\, \theta\big(d_{m+1}(\{\hat p, \hat f\}_{m+1}) < d_{\rm ini} < d_m(\{\hat p, \hat f\}_{m+1})\big) -
\end{aligned}
$$

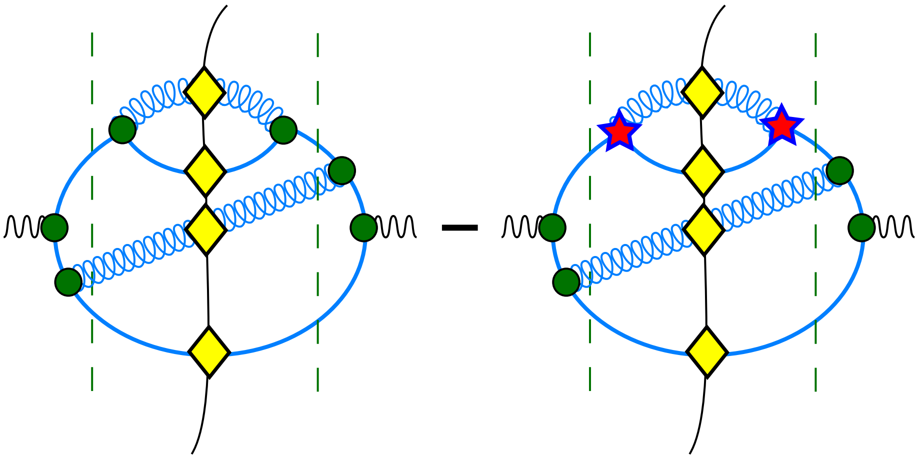

Figure 2: Illustration of $\sigma_3^{\rm R+S}$, eq. (8.43). As in figure 1, the three partons that cross the dash lines are resolvable at scale $d_{\rm ini}$. In the first graph, there is an additional perturbative splitting that is *not* resolvable at scale $d_{\rm ini}$. There is a subtraction graph in which the perturbative splitting is replaced by a splitting according to the dipole splitting formula, which is the order $\alpha_{\rm s}$ contribution to the splitting represented by the box in figure 1. The subtraction removes the soft/collinear divergence from the splitting. Finally, there is secondary showering represented in the figure by diamonds and in eq. (8.43) by the function $\tilde{I}$. There is a reweighting factor W that is not represented by a graphical symbol. This illustration is an attempt to depict the main idea behind eq. (8.43). Actually, though, the full matrix element $|\mathcal{M}\rangle$ to produce four partons appears in the first diagram and the full matrix element $|\mathcal{M}\rangle$ to produce three partons appears in the second diagram along with a sum over the dipole subtraction terms.

$$
\begin{aligned}
&- \sum_{\substack{i,j\\ \text{pairs}}} \sum_{k\neq i,j} \mathcal{D}_{ij,k}\big(\{\hat p,\hat f\}_{m+1}\big) \times \\
&\times\theta\big(\tilde d(\{p,f\}_m^{ij,k},\{l,y,z\}_{ij,k}) < d_{\rm ini} < d_m(\{p,f\}_m^{ij,k})\big)\Big\} \times \\
&\times\big[1+\mathcal{O}(\alpha_{\rm s})+\mathcal{O}(1\,{\rm GeV}/\sqrt{s})\big]\,. \qquad (8.44)
\end{aligned}
$$

Now if we add $\sigma_m^{\rm R+S}$ to $\sigma_{m,\Delta}^{\rm R+S}$ from the previous section, eq. (8.42), we get

$$
\begin{aligned}
\sigma_m^{\rm R+S} + \sigma_{m,\Delta}^{\rm B+S} = {}& \frac{1}{(m+1)!}\sum_{\{\hat f\}_{m+1}} \int d\Gamma(\{\hat p\}_{m+1}) \times \\
&\times\Bigg[\big\langle \mathcal{M}(\{\hat p,\hat f\}_{m+1})\big|\mathcal{M}(\{\hat p,\hat f\}_{m+1})\big\rangle\ F_{m+1}(\{\hat p\}_{m+1}) \times \\
&\quad\times\theta(d_{m+1}(\{\hat p,\hat f\}_{m+1}) < d_{\rm ini} < d_m(\{\hat p,\hat f\}_{m+1})) - \\
&\quad- \sum_{\substack{i,j\\ \text{pairs}}} \sum_{k\neq i,j} \mathcal{D}_{ij,k}(\{\hat p,\hat f\}_{m+1})\, F_m(\{p\}_m^{ij,k}) \times \\
&\quad\times\theta\big(\tilde d(\{p,f\}_m^{ij,k},\{l,y,z\}_{ij,k}) < d_{\rm ini} < d_m(\{p,f\}_m^{ij,k})\big)\Bigg] \times \\
&\times\Big[1+\mathcal{O}(\alpha_{\rm s})+\mathcal{O}\left(1\,{\rm GeV}/\sqrt{s}\right)\Big]\,. \qquad (8.45)
\end{aligned}
$$

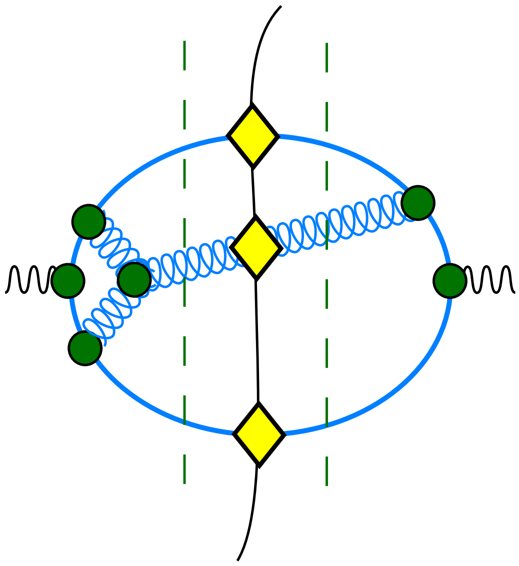

Figure 3: Illustration of $\sigma_3^{\mathrm{V+S}}$, eq. (8.47). As in figure 1, the three partons that cross the dash lines are resolvable at scale d_{ini}. To the left of the first dashed line, we illustrate one contribution to the one loop amplitude $|\mathcal{M}^{(1)}\rangle$. The infrared poles of $|\mathcal{M}^{(1)}\rangle$ are subtracted according to the dipole subtraction scheme. To the right, we illustrate one contribution to $\langle\mathcal{M}|$. All of this is included in eq. (8.47) in the function V, which is described in section 12. Finally, there is secondary showering represented in the figure by diamonds and in eq. (8.47) by the function $\tilde{I}$. There is a reweighting factor W that is not represented by a graphical symbol and there are two additional terms in eq. (8.47) that are not illustrated in the figure.

Comparing with eq. (6.3), we see that

$$\sigma_m^{\mathrm{R+S}} + \sigma_{m,\Delta}^{\mathrm{B+S}} = \sigma_m^{\mathrm{R-A}} \times \left[1 + \mathcal{O}(\alpha_{\mathrm{s}}) + \mathcal{O}\left(1\,\mathrm{GeV}/\sqrt{s}\right)\right]. \tag{8.46}$$

8.8 NLO virtual corrections with shower

In this subsection, we turn to the virtual loop corrections. We define[5]

$$\begin{aligned}\sigma_m^{\mathrm{V+S}} = \frac{1}{m!}\sum_{\{f\}_m}\int d\Gamma(\{p\}_m)\,\theta\big(d_{\mathrm{ini}} < d_m(\{p,f\}_m)\big)\,\tilde{I}(\{p,f\}_m)\,W_m^{\mathrm{V+S}}(\{p,f\}_m)\,\times \\ \times\Big\{V(\{p,f\}_m) - \sum_l\sum_{k\neq l} C_{l,k}(\{p\}_m, d_{\mathrm{ini}}) - \\ - \frac{\alpha_{\mathrm{s}}(\mu_{\mathrm{R}})}{2\pi}\,W_m^{(1)}(\{p,f\}_m)\,\big\langle\mathcal{M}(\{p,f\}_m)\big|\mathcal{M}(\{p,f\}_m)\big\rangle\Big\}.\end{aligned} \tag{8.47}$$

This formula is illustrated in figure 3. We integrate over the phase space for m partons with a cut on $d_m(\{p,f\}_m)$. There is a Monte Carlo interface function $\tilde{I}$ with the property (8.19). There is a reweighting factor $W_m^{\mathrm{V+S}}(\{p,f\}_m)$, which we discuss in section 13, eq. (13.19). This function has a perturbative expansion that begins with 1: $W_m^{\mathrm{V+S}}(\{p,f\}_m) = 1 + \mathcal{O}(\alpha_{\mathrm{s}})$. Then there is a factor with three terms. The first two contain the functions V and $C_{l,k}$

[5]The expression in eq. (8.47) can be simplified using eq. (12.8).

from the perturbative virtual loop contribution, eq. (6.6). The third contains the first order contribution $W_m^{(1)}$ to the Sudakov factor and is similar in structure to $\sigma^{\rm B+S}_{m,\{T,1\}}$, eq. (8.33).

To find the perturbative expansion of this, we replace $\tilde{I}$ by the jet observable function F, using the property (8.19) of $\tilde{I}$ and, using $W_m^{\rm V+S}(\{p,f\}_m) = 1 + \mathcal{O}(\alpha_{\rm s})$, we replace W by 1. This gives

$$\begin{aligned}
\sigma_m^{\rm V+S} = {}& \frac{1}{m!}\sum_{\{f\}_m}\int d\Gamma(\{p\}_m)\,\theta\big(d_{\rm ini} < d_m(\{p,f\}_m)\big)\,F_m(\{p\}_m)\times \\
&\times\Big\{V(\{p,f\}_m) - \sum_l\sum_{k\neq l} C_{l,k}(\{p\}_m, d_{\rm ini}) - \\
&\quad - \frac{\alpha_{\rm s}(\mu_{\rm R})}{2\pi}\,W_m^{(1)}(\{p,f\}_m)\,\big\langle\mathcal{M}(\{p,f\}_m)\big|\mathcal{M}(\{p,f\}_m)\big\rangle\Big\}\times \\
&\times\Big[1+\mathcal{O}(\alpha_{\rm s}) + \mathcal{O}\left(1\,{\rm GeV}/\sqrt{s}\right)\Big]\,.
\end{aligned}\tag{8.48}$$

We now add $\sigma^{\rm B+S}_{m,\{T,1\}}$ from eq. (8.33),

$$\begin{aligned}
\sigma_m^{\rm V+S} + \sigma^{\rm B+S}_{m,\{T,1\}} = {}& \frac{1}{m!}\sum_{\{f\}_m}\int d\Gamma(\{p\}_m)\,\theta\big(d_{\rm ini} < d_m(\{p,f\}_m)\big)F_m(\{p\}_m)\times \\
&\times\Big\{V(\{p,f\}_m) - \sum_l\sum_{k\neq l} C_{l,k}(\{p\}_m, d_{\rm ini})\Big\}\times \\
&\times\Big[1+\mathcal{O}(\alpha_{\rm s}) + \mathcal{O}\left(1\,{\rm GeV}/\sqrt{s}\right)\Big]\,.
\end{aligned}\tag{8.49}$$

Comparing with eq. (6.6), we see that

$$\sigma_m^{\rm V+S} + \sigma^{\rm B+S}_{m,\{T,1\}} = \sigma_m^{\rm V+A}\,\Big[1+\mathcal{O}(\alpha_{\rm s}) + \mathcal{O}\left(1\,{\rm GeV}/\sqrt{s}\right)\Big]\,. \tag{8.50}$$

8.9 Net result

Combining the contributions $\sigma^{\rm B+S}_{m,\{T,0\}}$ from eq. (8.32), $\sigma_m^{\rm R+S} + \sigma^{\rm B+S}_{m,\Delta}$ from eq. (8.46), and $\sigma_m^{\rm V+S} + \sigma^{\rm B+S}_{m,\{T,1\}}$ from eq. (8.50) we see that the calculation of σ_m with showers, using both the Born term and the correction terms, reproduces σ_m to next-to-leading order accuracy,

$$\sigma_m^{\rm B+S} + \sigma_m^{\rm R+S} + \sigma_m^{\rm V+S} = \sigma_m^{\rm B} + \sigma_m^{\rm R-A} + \sigma_m^{\rm V+A} + \mathcal{O}(\alpha_{\rm s}^{B_m+2}) + \mathcal{O}(\alpha_{\rm s}^{B_m}\times 1\,{\rm GeV}/\sqrt{s}\,)\,. \tag{8.51}$$

This is the result claimed in eq. (7.2).

We also note that the calculation of σ_m with showers using only the Born term reproduces σ_m to leading order accuracy,

$$\sigma_m^{\rm B+S} = \sigma_m^{\rm B} + \mathcal{O}(\alpha_{\rm s}^{B_m+1}) + \mathcal{O}(\alpha_{\rm s}^{B_m}\times 1\,{\rm GeV}/\sqrt{s}\,)\,. \tag{8.52}$$

9. Alternative LO partial cross sections with showers

In eq. (7.1), we imagined that the perturbative partial cross sections σ_m are known at order $\alpha_{\rm s}^{B_m+1}$ but that beyond a certain value $m_{\rm NLO}$ of m only order $\alpha_{\rm s}^{B_m}$ results are known.

Furthermore, we suppose that exact order $\alpha_s^{B_m}$ results are known only up to certain value $m_{\max}$ of m. We needed a construction of the corresponding contributions $\sigma_m^{\rm B+S}$ including showers. Our choice was to simply define $\sigma_m^{\rm B+S}$ for $m > m_{\rm NLO}$ according to eq. (8.1). This reproduces σ_m to leading order.

In this section, we wish to point out an alternative. For $m > m_{\rm NLO}$ one could replace $\sigma_m^{\rm B+S}$ by $\tilde\sigma_m^{\rm B+S}$ defined by

$$\begin{aligned}\tilde\sigma_m^{\rm B+S} = \frac{1}{m!}\sum_{\{f\}_m}\int d\Gamma(\{p\}_m)\,\theta\big(d_{\rm ini} < d_m(\{p,f\}_m)\big)\,W_m(\{p,f\}_m)\,\times \\ \times\big\langle\mathcal{M}(\{p,f\}_m)\big|\mathcal{M}(\{p,f\}_m)\big\rangle\,\tilde I(\{p,f\}_m)\,. \end{aligned} \tag{9.1}$$

Here we use the m-parton Born cross section modified by the reweighting factor W with a cut $d_{\rm ini} < d_m(\{p,f\}_m)$. We omit the dipole style splitting that we used in eq. (8.1). Instead, we pass the partonic state $\{p,f\}_m$ to the shower Monte Carlo program, represented here by the Monte Carlo interface function $\tilde I$ with the property eq. (8.19), as used already in eqs. (8.43) and (8.47). The shower Monte Carlo program generates parton showers from the scale $d_{\rm ini}$ downwards.

This is to say that for $m > m_{\rm NLO}$ we simply use the scheme of ref. [11]. This is perhaps a little simpler than the use of a splitting based on the dipole subtraction scheme for the first splitting.

There is a largest value, $m_{\max}$, of m such that we have available exact leading order matrix elements. Thus the sum in eq. (7.1) stops with $\sigma_m^{\rm B+S}$ or $\tilde\sigma_m^{\rm B+S}$ for $m = m_{\max}$. This leaves us with a zero cross section for producing more than $m_{\max}$ jets that are resolvable at scale $d_{\rm ini}$. Even though we lack the matrix elements for calculating exactly at leading order the cross section for producing more than $m_{\max}$ jets, we can still have an approximate cross section for producing many resolvable jets. Assuming that we are using $\tilde\sigma_m^{\rm B+S}$, all that we need to do is make a simple modification in the formula (9.1) for $\tilde\sigma_m^{\rm B+S}$ for $m = m_{\max}$: in $W_m(\{p,f\}_m)$ and in $\tilde I(\{p,f\}_m)$ we should replace $d_{\rm ini}$ by $d_m(\{p,f\}_m)$. That is, there are splittings that produce $m_{\max}$ partons according to the exact matrix element, with the corresponding Sudakov factors and α_s factors, represented in the modified $W_m(\{p,f\}_m)$. We remove from $W_m(\{p,f\}_m)$ the factor representing the probability that there was no splitting between $d_m(\{p,f\}_m)$ and $d_{\rm ini}$ because we now allow such splittings as part of the modified $\tilde I(\{p,f\}_m)$, which describes an indefinite number of splittings at scales below the scale $d_m(\{p,f\}_m)$ of this last splitting according to the exact matrix element.

10. The kinematics of parton splitting

In this section, we review the dipole splitting construction of Catani and Seymour [12], using the notation adopted for this paper. This covers one parton splitting into two partons (with the participation of a spectator parton). With a trivial extension that we have already described in section 3, this construction also covers one of m partons splitting to create an $m+1$-parton state.

The construction and deconstruction of dipole splitting outlined in section 3 is based on a transformation

$$\{\hat{p}_i, \hat{p}_j, \hat{p}_k\} \leftrightarrow \{\tilde{p}_{ij}, \tilde{p}_k, y, z, \phi\}\,. \tag{10.1}$$

The transformation from left to right combines partons i and j. With the inclusion of the rather trivial transformation to combine the flavors, this transformation was called $\mathcal{Q}_{ij,k}$ in eq. (3.8). The transformation from right to left splits parton ij and, with the inclusion of the flavor splitting, was called $\mathcal{R}_{ij}$ (or $\mathcal{R}_l$) in eq. (3.7).

Combining parton momenta works as follows. Starting with three massless momenta $\{\hat{p}_i, \hat{p}_j, \hat{p}_k\}$, Catani and Seymour define $\tilde{p}_{ij}$ and $\tilde{p}_k$ by

$$\begin{aligned}\tilde{p}_{ij} &= \hat{p}_i + \hat{p}_j - \frac{\hat{p}_i \cdot \hat{p}_j}{(\hat{p}_i + \hat{p}_j)\cdot \hat{p}_k}\,\hat{p}_k\,,\\ \tilde{p}_k &= \hat{p}_k + \frac{\hat{p}_i \cdot \hat{p}_j}{(\hat{p}_i + \hat{p}_j)\cdot \hat{p}_k}\,\hat{p}_k\,.\end{aligned} \tag{10.2}$$

Then it is evident that

$$\tilde{p}_{ij} + \tilde{p}_k = \hat{p}_i + \hat{p}_j + \hat{p}_k \tag{10.3}$$

and that

$$\tilde{p}_{ij}^2 = \tilde{p}_k^2 = 0\,. \tag{10.4}$$

The splitting parameter y is defined by

$$y = \frac{\hat{p}_i \cdot \hat{p}_j}{\hat{p}_i \cdot \hat{p}_j + (\hat{p}_i + \hat{p}_j)\cdot \hat{p}_k}\,, \tag{10.5}$$

so that

$$\begin{aligned}\tilde{p}_{ij} &= \hat{p}_i + \hat{p}_j - \frac{y}{1-y}\,\hat{p}_k\,,\\ \tilde{p}_k &= \frac{1}{1-y}\,\hat{p}_k\,.\end{aligned} \tag{10.6}$$

The splitting parameter z is defined by

$$z = \frac{\hat{p}_i \cdot \hat{p}_k}{(\hat{p}_i + \hat{p}_j)\cdot \hat{p}_k}\,. \tag{10.7}$$

Then $(1-z)$ is given by the same expression as for z but with $i \leftrightarrow j$. Alternative formulas for y and z are

$$y = \frac{\hat{p}_i \cdot \hat{p}_j}{\tilde{p}_{ij} \cdot \tilde{p}_k}\,, \qquad z = \frac{\hat{p}_i \cdot \tilde{p}_k}{\tilde{p}_{ij} \cdot \tilde{p}_k}\,. \tag{10.8}$$

To define the azimuthal angle ϕ we first define $k_\perp$ to be the part of $(\hat{p}_i - \hat{p}_j)/2$ that is orthogonal to both $\tilde{p}_{ij}$ and $\tilde{p}_k$,

$$k_\perp = \frac{1}{2}\,(\hat{p}_i - \hat{p}_j) - \left(z - \frac{1}{2}\right)\tilde{p}_{ij} + \left(z - \frac{1}{2}\right) y\,\tilde{p}_k\,. \tag{10.9}$$

One can show that $k_\perp$ obeys $k_\perp \cdot \tilde{p}_{ij} = 0$ and $k_\perp \cdot \tilde{p}_k = 0$ using the easily proved relations $(\hat{p}_i - \hat{p}_j)\cdot\tilde{p}_k = (2z-1)\,\tilde{p}_{ij}\cdot\tilde{p}_k$ and $(\hat{p}_i - \hat{p}_j)\cdot\tilde{p}_{ij} = (2z-1)\,y\,\tilde{p}_{ij}\cdot\tilde{p}_k$. A convenient alternative formula for $k_\perp$ is

$$k_\perp = z p_i - (1-z) p_j - (2z-1)\,\tilde{p}_{ij}\,. \tag{10.10}$$

The squared length of $k_\perp$ is

$$k_\perp^2 = -2y\,z(1-z)\,\tilde{p}_{ij}\cdot\tilde{p}_k\,. \tag{10.11}$$

The unit vector

$$\kappa_\perp = \frac{k_\perp}{\sqrt{-k_\perp^2}} \tag{10.12}$$

defines the plane of the splitting in a reference frame in which the vector parts of $\tilde{p}_{ij}$ and $\tilde{p}_k$ lie along the positive and negative z-axis, respectively. The azimuthal angle of $\kappa_\perp$ with respect to some convenient reference direction is ϕ . Stated perhaps more precisely, we say that one specifies ϕ as a shorthand for saying that one specifies the unit vector $\kappa_\perp$. Integrating over ϕ means integrating over $\kappa_\perp$ subject to the conditions that it is a unit vector orthogonal to $\tilde{p}_{ij}$ and $\tilde{p}_k$.

The inverse transformation, giving $\{\hat{p}_i, \hat{p}_j, \hat{p}_k\}$ in terms of $\{\tilde{p}_{ij}, \tilde{p}_k, y, z, \phi\}$, is easily obtained by combining eq. (10.9) and eq. (10.6),

$$\begin{aligned}
\hat{p}_i &= z\tilde{p}_{ij} + y(1-z)\tilde{p}_k + [2y\,z(1-z)\,\tilde{p}_{ij}\cdot\tilde{p}_k]^{1/2}\,\kappa_\perp\,,\\
\hat{p}_j &= (1-z)\tilde{p}_{ij} + yz\tilde{p}_k - [2y\,z(1-z)\,\tilde{p}_{ij}\cdot\tilde{p}_k]^{1/2}\,\kappa_\perp\,,\\
\hat{p}_k &= (1-y)\,\tilde{p}_k\,.
\end{aligned} \tag{10.13}$$

It is reasonably straightforward to work out the jacobian for this transformation. Defining

$$d\mathit{\Gamma}^{(n)}(\{p\}_n, Q) = \prod_{l=1}^{n}\Big((2\pi)^{-3} d^4 p_l\ \delta_+(p_l^2)\Big)(2\pi)^4\delta^4\left(\sum_{l=1}^{n} p_l - Q\right), \tag{10.14}$$

one finds [12]

$$\begin{aligned}
d\mathit{\Gamma}^{(3)}(\hat{p}_i, \hat{p}_j, \hat{p}_k; Q) &= d\mathit{\Gamma}^{(2)}(\tilde{p}_{ij}, \tilde{p}_k; Q)dy\,dz\,\frac{d\phi}{2\pi}\,\frac{2\tilde{p}_{ij}\cdot\tilde{p}_k}{16\pi^2}(1-y)\,\times\\
&\quad\times\Theta(z(1-z)>0)\,\Theta(y(1-y)>0)\,,
\end{aligned} \tag{10.15}$$

or, equivalently,

$$\begin{aligned}
d\mathit{\Gamma}^{(3)}(\hat{p}_i, \hat{p}_j, \hat{p}_k; Q)\,\frac{16\pi^2}{2\hat{p}_i\cdot\hat{p}_j} &= d\mathit{\Gamma}^{(2)}(\tilde{p}_{ij}, \tilde{p}_k; Q)\,dy\,dz\,\frac{d\phi}{2\pi}\frac{1-y}{y}\,\times\\
&\quad\times\Theta(z(1-z)>0)\,\Theta(y(1-y)>0)\,.
\end{aligned} \tag{10.16}$$

This transformation between two parton momenta plus splitting variables and three parton momenta is trivially extended to a map between m parton momenta plus splitting variables and $m+1$ parton momenta, as described in section 3. If the parton that splits is labelled l, the jacobian is

$$d\mathit{\Gamma}(\{p\}_m)\,dy_l\,\frac{1-y_l}{y_l}dz_l\,\frac{d\phi_l}{2\pi} = d\mathit{\Gamma}(\{\hat{p}\}_{m+1})\,\frac{16\pi^2}{2\hat{p}_{l,1}\cdot\hat{p}_{l,2}}\,. \tag{10.17}$$

In our applications, this result will appear with symmetry factors and sums over the indices of splitting partons. Then it looks like

$$
\begin{aligned}
&\frac{1}{m!}\sum_{\{f\}_m}\int d\Gamma(\{p\}_m)\sum_{l=1}^{m}\sum_{k\neq l}\int_0^1\frac{dy_l}{y_l}\int_0^1 dz_l\int_0^{2\pi}\frac{d\phi_l}{2\pi}\frac{1}{2}\sum_{\hat f_{l,1},\hat f_{l,2}}\delta^{f_l}_{\hat f_{l,1}+\hat f_{l,2}}(1-y_l)h(\{\hat p,\hat f\}_{m+1}) = \\
&\quad = \frac{1}{(m+1)!}\sum_{\{f\}_{m+1}}\int d\Gamma(\{\hat p\}_{m+1})\sum_{\substack{i,j\\ \text{pairs}}}\sum_{k\neq i,j}\frac{16\pi^2}{2\hat p_i\cdot\hat p_j}\,h(\{\hat p,\hat f\}_{m+1})\,,
\end{aligned}
\tag{10.18}
$$

where $h(\{\hat p,\hat f\}_{m+1})$ represents any well behaved function. To prove this, note that the left hand side has symmetry factors $1/m!$ and $1/2$ and a sum over m values of the index l and $(m-1)$ values of the index k. On the right hand side we have a symmetry factor $1/(m+1)!$ and a sum over $(m+1)m/2$ values of the index pair i,j and $(m-1)$ values of the index k. If we put the sums over parton indices outside the integrals, each term is the same, so we can just take one term and multiply by the number of terms. We note that the net counting factor $m(m-1)/[2m!]$ on the left hand side equals the net counting factor $(m+1)m(m-1)/[2(m+1)!]$ on the right hand side. Thus eq. (10.17) implies eq. (10.18).

We now examine the relation between parton splitting and the jet resolution functions d_{ij} defined in eq. (5.3) and $\tilde d$ defined in eq. (6.4). Taking massless parton momenta in eq. (5.3) and using eq. (10.8), we have

$$
d_{ij}\equiv d(\hat p_i,\hat p_j)=\frac{2\hat p_i\cdot\hat p_j}{s}\,\min\left\{\frac{\hat p_i\cdot n}{\hat p_j\cdot n},\frac{\hat p_j\cdot n}{\hat p_i\cdot n}\right\}=\frac{2\tilde p_{ij}\cdot\tilde p_k}{s}\,y\,\min\left\{\frac{\hat p_i\cdot n}{\hat p_j\cdot n},\frac{\hat p_j\cdot n}{\hat p_i\cdot n}\right\},
\tag{10.19}
$$

where n is a unit vector that defines the time axis in the e^+e^- c.m. frame. For $\hat p_i\cdot n$ and $\hat p_j\cdot n$, we can use eq. (10.13). Of particular interest is the case $0<y\ll z<1$, in which $\hat p_i$ is either collinear to $\tilde p_{ij}$ or both soft and collinear. Under this condition we have

$$
d_{ij}\sim\frac{2\tilde p_{ij}\cdot\tilde p_k}{s}\,y\,\min\left\{\frac{z}{1-z},\frac{1-z}{z}\right\}\equiv\frac{2\tilde p_{ij}\cdot\tilde p_k}{s_{ij}}\,\tilde d\,.
\tag{10.20}
$$

Here $\tilde d$ is the function defined eq. (6.4) (with $y_l\to y$, $z_l\to z$ and with the scale s_l for parton l renamed to s_{ij}). We used this function to limit parton splitting in eq. (6.3) and eq. (8.6): $\tilde d<d_{\rm ini}$. We see that, under these conditions, limiting $\tilde d$ limits d_{ij}, although there is a factor $2\tilde p_{ij}\cdot\tilde p_k/s_{ij}$ that relates the size of the two resolution measures.

Also if interest is the case $0<z\ll 1$, $0<y\ll 1$, $y/z\sim 1$, in which $\hat p_i$ is soft but not collinear to $\tilde p_{ij}$ or $\tilde p_k$. Under this condition, we have

$$
d_{ij}\sim\frac{2\tilde p_{ij}\cdot\tilde p_k}{s}\,y\,z\left\{1+\frac{y}{z}\frac{\tilde p_k\cdot n}{\tilde p_{ij}\cdot n}+\sqrt{\frac{y}{z}}\frac{[2\tilde p_{ij}\cdot\tilde p_k]^{1/2}\;\kappa_\perp\cdot n}{\tilde p_{ij}\cdot n}\right\}.
\tag{10.21}
$$

Thus in the soft limit, d_{ij} has the same scaling behavior as $\tilde d$, i.e. a factor yz, but the ratio of these functions is not generally 1 and depends on the ratio of y to z.

We should also be concerned about the region $0 < z \ll y < 1$. This region is not enhanced in the integration over y and z because the relevant splitting functions in eqs. (6.3) and (8.6) are not singular for $z \to 0$ at fixed y. Nevertheless, since $\tilde{d} \ll d_{ij}$ in this region, we should understand what the restriction $\tilde{d} \ll d_{\rm ini}$ means. Consider, then, the resolution parameter for combining parton i with the spectator parton k. Using the definition (5.3) together with eq. (10.8), we have

$$d_{ik} \equiv d(\hat{p}_i, \hat{p}_k) = \frac{2\tilde{p}_{ij} \cdot \tilde{p}_k}{s}\, z\,(1-y) \min\left\{ \frac{\hat{p}_i \cdot n}{\hat{p}_k \cdot n}, \frac{\hat{p}_k \cdot n}{\hat{p}_i \cdot n} \right\}. \tag{10.22}$$

Under the condition $0 < z \ll y < 1$, this is

$$d_{ik} \sim \frac{2\tilde{p}_{ij} \cdot \tilde{p}_k}{s}\, y\, z \min\left\{ 1, \frac{(1-y)^2}{y^2} \right\} < \frac{2\tilde{p}_{ij} \cdot \tilde{p}_k}{s_{ij}}\, \tilde{d}. \tag{10.23}$$

Thus when $z \ll y$ the condition $\tilde{d} < d_{\rm ini}$ restricts parton i to be close to the spectator parton k.

11. Splitting functions

In this section we review the functions $\mathcal{D}_{ij,k}$ used in the dipole subtraction algorithm [12]. We then specify the functions $\boldsymbol{S}_l(p, f, Y)$ and $\big\langle \boldsymbol{S}(y, z, f) \big\rangle$ that we use to describe parton splitting and that are derived from the $\mathcal{D}_{ij,k}$ functions.

We begin with the functions $\mathcal{D}_{ij,k}$ that form the basis for the dipole subtraction algorithm. The basic idea is that the squared matrix element $\langle \mathcal{M}(\{\hat{p}\}_{m+1}) | \mathcal{M}(\{\hat{p}\}_{m+1}) \rangle$ for $m+1$ partons has a potential singularity that could lead to a divergent integral when the dot product of any pair of the momenta, say $p_i \cdot p_j$, goes to zero. The matrix element takes a rather simple form in this limit. In fact, the form would be extremely simple were it not for the fact that one has $p_i \cdot p_j \to 0$ not only when p_i becomes collinear with p_j but also when $p_i \to 0$ with p_j fixed and p_i not necessarily collinear with p_j (or when p_j becomes soft with fixed p_i). This configuration can lead to a divergent integral if the soft parton i (or j) is a gluon. When the soft gluon i couples to parton j and another parton k, the structure of the limiting function depends on what k is. For this reason, Catani and Seymour write the matrix element in the limit $p_i \cdot p_j \to 0$ as a sum of terms labelled by the index k of a "spectator" parton. That is

$$\big\langle \mathcal{M}\big(\{\hat{p}, \hat{f}\}_{m+1}\big) \big| \mathcal{M}\big(\{\hat{p}, \hat{f}\}_{m+1}\big) \big\rangle = \sum_{k \neq i,j} \mathcal{D}_{ij,k}(\{\hat{p}, \hat{f}\}_{m+1}) + \cdots, \tag{11.1}$$

where the dots stand for terms that are nonsingular in the limit $p_i \cdot p_j \to 0$. The dipole functions $\mathcal{D}_{ij,k}$ have a simple structure of the form that we reviewed in eq. (4.5),

$$\mathcal{D}_{ij,k}(\{\hat{p}, \hat{f}\}_{m+1}) = \frac{1}{2\hat{p}_i \cdot \hat{p}_j} \big\langle \mathcal{M}(\{p, f\}_m^{ij,k}) \big| \frac{\boldsymbol{T}_{ij} \cdot \boldsymbol{T}_k}{-\boldsymbol{T}_{ij}^2} \boldsymbol{V}_{ij}(\{\tilde{p}_{ij}, y, z, \phi\}_{ij,k}, f_i, f_j) \big| \mathcal{M}(\{p, f\}_m^{ij,k}) \big\rangle. \tag{11.2}$$

First, there is a singular factor $1/(2\hat{p}_i \cdot \hat{p}_j)$. Then there is the Born amplitude $|\mathcal{M}(\{p, f\}_m^{ij,k})\rangle$ and its complex conjugate for m parton momenta and flavors formed by combining partons

i and j with the help of the spectator k according to the formulas of the previous section. These amplitudes are vectors in the color and spin space of the partons. More precisely, for each of the m partons there is a spin space spanned by two basis vectors $|s\rangle$ and a color space spanned by three basis vectors $|c\rangle$ in the case of a quark or antiquark or eight basis vectors $|c\rangle$ in the case of a gluon. The amplitude $|\mathcal{M}(\{p,f\}_m)\rangle$ lies in the direct product of the m spin spaces and m color spaces. In eq. (11.2) there are color and spin operators that act on given single parton factors in the direct product space, with the parton factor affected labelled by a subscript ij (for the parton obtained by combining partons i and j) or k (for the spectator parton).

For the color, there is an operator, the SU(3) generator $\boldsymbol{T}_{ij}^a$, that acts on the color space for parton ij and there is another SU(3) generator $\boldsymbol{T}_k^a$ that acts on the color space for the spectator parton k. The dot product indicates a sum over a from 1 to 8. In the denominator there is a factor $\boldsymbol{T}_{ij}\cdot\boldsymbol{T}_{ij}$, which is simply a number equal to $C_{\rm A}$ if parton ij is a gluon and $C_{\rm F}$ if it is a quark.

The remaining factor, $\boldsymbol{V}_{ij}$, is a function of variables defined by considering that one combines partons i and j with the help of the spectator k, as in eqs. (3.8), (3.9) and (4.4). Specifically, $\boldsymbol{V}_{ij}$ is an operator on the spin space of the mother parton before the splitting and depends on the momentum $\tilde{p}_{ij}$ of this parton. It is also a function of the splitting variables $\{y,z,\phi\}$ and the daughter flavors $\{f_i,f_j\}$.

The shower algorithm of this paper makes use of splitting functions $\boldsymbol{S}_l(p,Y)$. We found it useful to take these functions to be proportional to the functions $\boldsymbol{V}_{ij}$, using eq. (8.41),

$$\frac{\alpha_{\rm s}}{2\pi}\,\frac{16\pi^2}{1-y}\,\boldsymbol{S}_l(p,f,Y) = \boldsymbol{V}_l(\{p,y,z,\phi\},f_1,f_2)\,, \tag{11.3}$$

where $Y=\{y,z,\phi,f_1,f_2\}$. (Note that the subscript l does not enter into the functional dependence of these functions but merely tells on what parton's spin space the operator acts.) Since we use mostly the functions $\boldsymbol{S}_l(p,f,Y)$, we present here the standard definition of $\boldsymbol{V}_{ij}$ simply translated into the new notation.

The definition begins by separating the possibilities for flavors,

$$\begin{aligned}\boldsymbol{S}_l(p,f,Y) = {}& \delta_{gf_1}\delta_{gf_2}\boldsymbol{S}_{gg}(p,z,y,\kappa_\perp) + \\ &+\sum_{r=u,\bar{u},d,\bar{d},\dots}\Big[\delta_{rf_1}\delta_{\bar{r}f_2}\boldsymbol{S}_{q\bar{q}}(p,z,y,\kappa_\perp) + \\ &\qquad + \delta_{rf_1}\delta_{gf_2}\boldsymbol{S}_{qg}(z,y) + \delta_{gf_1}\delta_{rf_2}\boldsymbol{S}_{qg}(1-z,y)\Big]\,. \end{aligned} \tag{11.4}$$

Then for the splitting of a quark or antiquark into the same flavor quark or antiquark plus a gluon the splitting function is

$$\langle s|\boldsymbol{S}_{qg}(z,y)|s'\rangle = C_{\rm F}\,(1-y)\left[\frac{2}{1-z(1-y)}-(1+z)\right]\delta_{ss'}\,, \tag{11.5}$$

where the s and s' are the spin indices of the emitter quark or antiquark. For the splitting of a gluon into a quark and an antiquark, we again denote the spin indices of the emitter gluon by s and s' and define

$$\langle s|\boldsymbol{S}_{q\bar{q}}(p,z,y,\kappa_\perp)|s'\rangle = T_{\rm R}\,(1-y)\,\epsilon^*_\mu(p,s)\left[-g^{\mu\nu}-4z(1-z)\,\kappa_\perp^\mu\kappa_\perp^\nu\right]\epsilon_\nu(p,s')\,. \tag{11.6}$$

Finally for the splitting of a gluon into two gluons the splitting function is

$$\begin{aligned}\langle s|\boldsymbol{S}_{gg}(p,z,y,\kappa_\perp)|s'\rangle \;=\; 2C_{\mathrm{A}}\,(1-y)\,\epsilon^*_\mu(p,s)\Bigg[-g^{\mu\nu}\bigg(&\frac{1}{1-z(1-y)}+\\ &+\frac{1}{1-(1-z)(1-y)}-2\bigg)+\\ &+2\,z(1-z)\,\kappa_\perp^\mu\kappa_\perp^\nu\Bigg]\epsilon_\nu(p,s')\,. \end{aligned}\tag{11.7}$$

In the gluon splitting functions, $\epsilon(p,s)$ is the polarization vector for the emitter gluon. With a change of the gauge used in defining $\epsilon(p,s)$, one has $\epsilon(p,s)\to\epsilon(p,s)+\lambda\,p$. However, the matrix elements are unchanged because $\epsilon(p,s)\cdot p=0$ and $\kappa_\perp\cdot p=0$.

It is also of interest to know whether these matrix elements depend on the spectator momentum, p_k. No spectator momentum appears explicitly in eqs. (11.6) and (11.7), but recall that $\kappa_\perp$ is a unit vector orthogonal to p and to p_k. If we change p_k to p'_k, we can define a new vector $\kappa'_\perp$ to specify the azimuthal angle of the splitting by

$$\kappa'_\perp = \kappa_\perp - \frac{p'_k\cdot\kappa_\perp}{p'_k\cdot p}\,p\,. \tag{11.8}$$

This new vector is still a unit vector, still orthogonal to p, but now is orthogonal to p'_k instead of p_k. The change does not affect the matrix element because $\epsilon(p,s)\cdot p=0$. This justifies the notation that $\boldsymbol{V}_{ij}$ depends on p and on the splitting variables (including ϕ or, equivalently, $\kappa_\perp$) but not on the spectator momentum p_k.

The Sudakov exponent in the showering formula contains the average over angle and flavors of $\boldsymbol{S}_l$, eq. (8.10),

$$\big\langle \boldsymbol{S}(y,z,f)\big\rangle \equiv \int\frac{d\phi}{2\pi}\,\frac{1}{2}\sum_{f_1,f_2}\delta^f_{f_1+f_2}\,\boldsymbol{S}_l(p,Y)\,. \tag{11.9}$$

Here the angular average is in 2 transverse dimensions. There are extra terms in $\langle\boldsymbol{S}\rangle$ if one works in $2-2\epsilon$ dimensions, but we do not need these terms in this paper. For the angular average, we can use

$$\int\frac{d\phi}{2\pi}\,\epsilon^*_\mu(p,s)\,\kappa_\perp^\mu\kappa_\perp^\nu\,\epsilon_\nu(p,s') = \frac{1}{2}\,\delta_{ss'}\,. \tag{11.10}$$

Then we see that $\big\langle\boldsymbol{S}_l(y,z,f)\big\rangle$ is a number times the unit operator on the partonic spin space. A simple calculation gives for a quark or antiquark emitter

$$\big\langle\boldsymbol{S}(y,z,f)\big\rangle = C_{\mathrm{F}}\,(1-y)\left[\frac{2}{1-z(1-y)}-(1+z)\right], \qquad f\in\{\mathrm{u},\bar{\mathrm{u}},\mathrm{d},\dots\}\,. \tag{11.11}$$

For a gluon emitter, including splittings into both gluon and quark-antiquark pairs, one finds

$$\begin{aligned}\big\langle\boldsymbol{S}(y,z,g)\big\rangle \;=\; &C_{\mathrm{A}}\,(1-y)\left[\frac{1}{1-z(1-y)}+\frac{1}{1-(1-z)(1-y)}-2+z(1-z)\right]+\\ &+N_{\mathrm{f}}\,T_{\mathrm{R}}\,(1-y)\,[1-2z(1-z)]\,. \end{aligned}\tag{11.12}$$

The $y \to 0$ limits of these functions match the Altarelli-Parisi splitting functions,

$$
\begin{aligned}
\big\langle \boldsymbol{S}(y,z,f) \big\rangle &\to C_{\rm F}\, \frac{1+z^2}{1-z} = P_{\rm q/q}(z)\,, \qquad f \in \{\rm u, \bar u, d, \dots\}\,, \\
\big\langle \boldsymbol{S}(y,z,g) \big\rangle &\to C_{\rm A}\, \frac{[1-z(1-z)]^2}{z(1-z)} + N_{\rm f}\, T_{\rm R}\, [1-2z(1-z)] \\
&= \frac{1}{2}\, P_{\rm g/g}(z) + N_{\rm f} P_{\rm q/g}(z)\,.
\end{aligned}
\tag{11.13}
$$

Note that some of the Altarelli-Parisi functions are singular as $z \to 0$ or $z \to 1$. However, for $y > 0$ the splitting functions $\boldsymbol{S}$ are not singular as $z \to 0, 1$.

12. Virtual contributions

In eq. (4.7), we specified the virtual loop contribution in the dipole subtraction scheme in terms of a function $V(\{p,f\}_m)$,

$$
\sigma^{\rm V+A} = \int_m \left[d\sigma^{\rm V} + \int_1 d\sigma^{\rm A} \right] = \frac{1}{m!} \sum_{\{f\}_m} \int d\Gamma(\{p\}_m) V(\{p,f\}_m) F_m(\{p\}_m)\,. \tag{12.1}
$$

The definition [12] is

$$
\begin{aligned}
V(\{p,f\}_m) = \Big[& 2\,{\rm Re}\, \big\langle \mathcal{M}(\{p\}_m) \big| \mathcal{M}^{(1)}(\{p\}_m;\epsilon) \big\rangle + \\
& + {\rm Re}\, \big\langle \mathcal{M}(\{p\}_m) \big| \boldsymbol{I}(\{p,f\}_m;\epsilon) \big| \mathcal{M}(\{p\}_m) \big\rangle \Big]_{\epsilon=0}\,.
\end{aligned}
\tag{12.2}
$$

Here $\big|\mathcal{M}^{(1)}(\{p\}_m;\epsilon)\big\rangle$ denotes the virtual one-loop correction to the m-parton matrix element in $4-2\epsilon$ dimensions. This matrix element is singular as $\epsilon \to 0$. The singularity is cancelled by the integral of $d\sigma^{\rm A}$, performed in $4-2\epsilon$ dimensions. The result of this integration is the Born amplitude $|\mathcal{M}(\{p\}_m)\rangle$ times an operator $\boldsymbol{I}(\epsilon)$ in the color space of the final-state partons,

$$
\boldsymbol{I}(\{p,f\}_m;\epsilon) = \frac{\alpha_{\rm s}}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \sum_{i=1}^{m} \mathcal{V}(f_i,\epsilon) \sum_{\substack{j=1 \\ j \neq i}}^{m} \frac{\boldsymbol{T}_i \cdot \boldsymbol{T}_j}{-\boldsymbol{T}_i^2} \left(\frac{4\pi\mu^2}{2p_i \cdot p_j} \right)^{\epsilon}. \tag{12.3}
$$

The function $\mathcal{V}(f,\epsilon)$ has a simple expansion about $\epsilon = 0$,

$$
\mathcal{V}(f,\epsilon) = C_f \left(\frac{1}{\epsilon^2} - \frac{\pi^2}{3} \right) + \gamma_f \left(\frac{1}{\epsilon} + 1 \right) + K_f + \mathcal{O}(\epsilon)\,, \tag{12.4}
$$

where C_f, γ_f, and K_f are given in eq. (2.7).

When we adapt the dipole subtraction scheme to the calculation of partial cross sections based on a cut $d_{\rm ini}$ on the "distance" between partons, we need a correction to the subtraction term for virtual graphs. The correction involves a function $C_{l,k}$ defined in eq. (6.7),

$$
C_{l,k}(\{p,f\}_m, d_{\rm ini}) = \int_0^1 \frac{dy}{y} \int_0^1 dz \int_0^{2\pi} \frac{d\phi}{2\pi}\, \frac{1}{2} \sum_{\hat f_1, \hat f_2} \delta^{f_l}_{\hat f_1 + \hat f_2}\, \theta\big(d_{\rm ini} < \tilde d(\{p,f\}_m, l, y, z)\big) \times
$$

$$\times \frac{y(1-y)\,2p_l\cdot p_k}{16\pi^2}\,\mathcal{D}_{l,k}(\{p,f\}_m;Y)\,. \tag{12.5}$$

Here the distance function $\tilde{d}(\{p,f\}_m,l,y,z)$ is defined in eq. (6.4). To find the explicit form of $C_{l,k}(\{p,f\}_m,d_{\rm ini})$, we use eqs. (11.2), (11.3) and (11.9) to relate the sum over daughter flavors and average over azimuthal angle ϕ of $\mathcal{D}_{l,k}$ to $\langle \boldsymbol{S}(y,z,f)\rangle$. This gives

$$C_{l,k}(\{p,f\}_m,d_{\rm ini}) = \frac{\alpha_{\rm s}}{2\pi}\,\mathcal{C}\!\left(\frac{s\,d_{\rm ini}}{s_l},f_l\right)\big\langle\mathcal{M}(\{p,f\}_m)\big|\frac{\boldsymbol{T}_l\cdot\boldsymbol{T}_k}{-\boldsymbol{T}_l^2}\big|\mathcal{M}(\{p,f\}_m)\big\rangle\,, \tag{12.6}$$

where

$$\frac{\alpha_{\rm s}}{2\pi}\,\mathcal{C}(x,f) = \int_0^1\frac{dy}{y}\int_0^{1/2}dz\,\theta\!\left(\frac{z}{(1-z)}\geq\frac{x}{y}\right)\big[\big\langle\boldsymbol{S}(y,z,f)\big\rangle+\big\langle\boldsymbol{S}(y,1-z,f)\big\rangle\big]\,. \tag{12.7}$$

When we sum over the spectator label k, eq. (12.6) simplifies to

$$\sum_{k\neq l} C_{l,k}(\{p,f\}_m,d_{\rm ini}) = \frac{\alpha_{\rm s}}{2\pi}\,\mathcal{C}\!\left(\frac{s\,d_{\rm ini}}{s_l},f_l\right)\big\langle\mathcal{M}(\{p,f\}_m)\big|\mathcal{M}(\{p,f\}_m)\big\rangle\,. \tag{12.8}$$

Using the explicit form of $\langle \boldsymbol{S}(y,z,f)\rangle$ given in eqs. (11.11) and (11.12), we can perform the integrations over y and z to obtain

$$\begin{aligned}\mathcal{C}\,(x,f) \;=\; & \frac{1}{2}\,C_f\log^2(x)+\gamma_f\log(4x)-2C_f\log(x)\log(1+x)-\frac{\pi^2}{6}\,C_f+\\ & +\gamma_f 2x\log(2x)-2\gamma_f(1+x)\log(1+x)+\gamma_f(1-x)-2C_f\,{\rm Li}_2(-x)+\\ & +\delta_{gf}(C_{\rm A}-2T_{\rm R}N_{\rm F})\left(\frac{1}{3}x\log\frac{2x}{1+x}+\frac{1+2x-3x^2}{12(1+x)}\right).\end{aligned} \tag{12.9}$$

13. Sudakov factors

The functions $\sigma_m^{\rm B+S}$, $\sigma_m^{\rm R+S}$, and $\sigma_m^{\rm V+S}$ contain reweighting functions W. These functions contain rather trivial ratios of $\alpha_{\rm s}$ at different scales and not so trivial Sudakov exponentials. In addition, the splitting operators $\boldsymbol{E}_{l,k}$ in $\sigma_m^{\rm B+S}$ contain Sudakov exponentials. In this section, we discuss these factors and, in particular, give definitions for the functions W that we use. The presentation also touches on the Sudakov exponentials in the the Monte Carlo interface functions I and $\tilde{I}$ in $\sigma_m^{\rm B+S}$, $\sigma_m^{\rm R+S}$, and $\sigma_m^{\rm V+S}$.

13.1 The standard NLL Sudakov factor and its interpretation

The formula (8.1) for $\sigma_m^{\rm B+S}$ contains a reweighting factor $W_m(\{p,f\}_m)$ consisting of a product of ratios of $\alpha_{\rm s}$ at different scales, eq. (8.2), and a Sudakov exponential. Following ref. [11], we base this factor on a construction of a splitting history corresponding to the final state $\{p,f\}_m$ using the "k_T" jet algorithm (slightly extended to include flavor information) as described in section 5. The algorithm combines the final state partons so as to produce a QCD tree graph for $e^+e^-\to\{p,f\}_m$. Let the lines in this tree graph be labeled by an index L. We can label the vertices according to the order in which the partons were combined. We start with m partons and combine two of them at a vertex with label $V=m$

to produce an $m-1$ parton state. Then we combine two of these $m-1$ partons at a vertex with label $V = m-1$ to produce an $m-2$ parton state. Finally, the last 2 partons are combined at vertex 2 to form a γ or Z. At each stage between vertices V and $V+1$ there are $n = V$ partons. The line labels L for these partons form a set that we can call $I(V)$ (for the intermediate state after vertex V).

Each vertex V has an associated distance measure $d(V)$, with $d(V) \geq d(V+1)$. The values of the $d(V)$ will appear in $W_m(\{p,f\}_m)$. Typically, $d(V)$ is the k_T measure (5.3) for the splitting at vertex V. In some cases, we have redefined $d(V)$ to equal $d(V+1)$.

With these definitions, for each line L in the splitting graph there is initial resolution parameter $d(V_{\rm i}(L))$ equal to $d(V)$ for the vertex $V_{\rm i}(L)$ at which the line originates and a final resolution parameter $d(V_{\rm f}(L))$ equal to $d(V)$ for the vertex $d(V_{\rm f}(L))$ at which the line splits. For a line that enters the final state, we take $d(V_{\rm f}(L)) \equiv d_{\rm ini}$. Thus $1 \geq d_{\rm i}(L) \geq d_{\rm f}(L) \geq d_{\rm ini}$. We denote the flavor of line L by $f(L)$.

With this notation, we can state the definition of the reweighting factor W as given in [11]. Briefly, the idea is the following. Consider the Born cross section $\sigma_m^{\rm B}$ with observation function $F_m(\{p\}_m) = 1$. This is the Born level cross section for making m jets, using resolution parameter $d_{\rm ini}$. By inserting a factor $W_m(\{p,f\}_m)$ into the integral for $\sigma_m^{\rm B}$, we obtain the approximation to the m-jet cross section that sums logarithms of $d_{\rm ini}$ at the leading log and next-to-leading log level. The required reweighting factor is a product of $\alpha_{\rm s}$ factors, one for each vertex, and a product of Sudakov factors, one for each line,

$$W_m^{\rm NLL}(\{p,f\}_m) = \prod_{V>2} \frac{\alpha_{\rm s}\big(\sqrt{d(V)s}\big)}{\alpha_{\rm s}(\mu_{\rm R})} \prod_{L} \frac{\Delta_{f(L)}(d(V_{\rm i}(L)), d_{\rm ini})}{\Delta_{f(L)}(d(V_{\rm f}(L)), d_{\rm ini})}\,, \tag{13.1}$$

where

$$\Delta_f(d_{\rm max}, d_{\rm min}) = \exp\left[-\int_{d_{\rm min}}^{d_{\rm max}} \frac{d\lambda}{\lambda}\,\frac{\alpha_{\rm s}(\sqrt{\lambda s})}{2\pi}\,\left\{C_f \log\left(\frac{d_{\rm max}}{\lambda}\right) - \gamma_f\right\}\right]. \tag{13.2}$$

The coefficients C_f and γ_f are given in eq. (2.7). Thus one multiplies the perturbative matrix element by Sudakov factors that would have been associated with the lines if the final state had been generated by a suitable parton shower algorithm according to the specified QCD tree graph with the restriction that no splittings were to be generated that were unresolvable at resolution parameter $d_{\rm ini}$. For each QCD vertex we also use the reweighting factor to replace $\alpha_{\rm s}(\mu_{\rm R})$ used in the perturbative matrix element by the running coupling used in a standard parton shower. Our default reweighting factor $W_m(\{p,f\}_m)$ will be a slightly modified version of $W_m^{\rm NLL}(\{p,f\}_m)$ that is equivalent if we neglect terms suppressed by a color factor $1/N_{\rm c}^2$. We define the modified version below after examining the structure of $W_m^{\rm NLL}(\{p,f\}_m)$.

The function $W_m^{\rm NLL}(\{p,f\}_m)$ as given by eq. (13.1) is a product of factors, one for each propagator in the jet-structure graph. There is another way to write it that focuses on the evolution from one value of $d(V)$ to the next (See figure 4).

To reorganize $W_m^{\rm NLL}(\{p,f\}_m)$, we define a special vertex $V_{\rm s}(L)$ corresponding to each line L in the graph. We can think of $V_{\rm s}(L)$ as the vertex where the history of line L

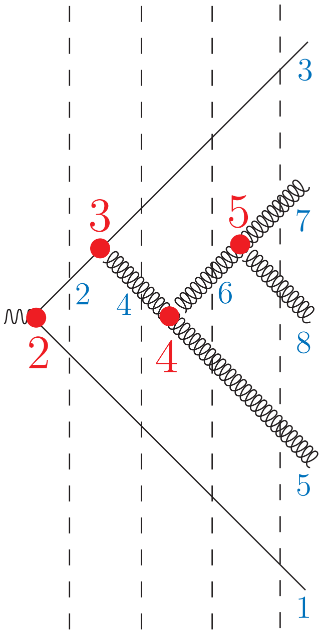

Figure 4: Illustration of the tree graph for a shower constructed from a five parton final state. The lines in the graph are numbered $L = 1, \ldots, 8$. The vertices are numbered $V = 2, \ldots, 5$. The resolution parameters $d(V)$ for the vertices decrease: $d(2) \geq d(3) \geq d(4) \geq d(5) > d_{\rm ini}$. There is a starting vertex $V_{\rm s}(L)$ for each parton line L. Here $V_{\rm s}(1) = V_{\rm s}(2) = V_{\rm s}(3) = 2$, $V_{\rm s}(4) = V_{\rm s}(5) = 3$, $V_{\rm s}(6) = V_{\rm s}(7) = 4$, and $V_{\rm s}(8) = 5$. We will write the Sudakov factor as a product of factors that represent the probabilities that the partons in each of the indicated intermediate state evolve from scale $d(V)$ to scale $d(V+1)$ without splitting. For the last final state, the Sudakov factor is the probability for the five partons to evolve from scale $d(5)$ to $d_{\rm ini}$ without splitting.

started (*cf.* section 3.2 of [11]). The identification of the $V_{\rm s}(L)$ is recursive. Consider the starting vertex $2 : \gamma/Z \to q(L_1)\bar{q}(L_2)$, where our notation indicates that 2 is the label of the vertex, L_1 is the label of the quark line and L_2 is the label of the antiquark line. We define $V_{\rm s}(L_1) = V_{\rm s}(L_2) = 2$. Then for any vertex $V_i : q(L_0) \to q(L_1)g(L_2)$, we define $V_{\rm s}(L)$ for the two daughter lines by $V_{\rm s}(L_1) = V(L_0)$ while $V_{\rm s}(L_2) = V_i$. Similarly for any vertex $V_i : \bar{q}(L_0) \to \bar{q}(L_1)g(L_2)$, we define $V_{\rm s}(L_1) = V_{\rm s}(L_0)$ while $V_{\rm s}(L_2) = V_i$. For a vertex $V_i : g(L_0) \to g(L_1)g(L_2)$, we first provide a definite labelling for the daughters by, say, letting L_1 be the label of the daughter gluon with splitting fraction $z_1 > 1/2$. Then we define $V_{\rm s}(L_1) = V(L_0)$ while $V_{\rm s}(L_2) = V_i$. Finally, for any vertex $V_i : g(L_0) \to q(L_1)\bar{q}(L_2)$, we define $V_{\rm s}(L_1) = V_{\rm s}(L_2) = V_{\rm s}(L_0)$.

In the case that L labels a quark or antiquark line, we define another special vertex $V_{\rm o}(L)$ that is the vertex where the quark line originated in the jet-structure graph: either the $\gamma/Z \to q\bar{q}$ vertex $V = 2$ or else a later $g \to q\bar{q}$ vertex V. (The subscript is "o" for "originated.")

The reader can check that with this definition

$$W_m^{\rm NLL}(\{p,f\}_m) = \prod_{V>2} \frac{\alpha_{\rm s}\big(\sqrt{d(V)s}\big)}{\alpha_{\rm s}\big(\mu_{\rm R}\big)} \prod_{V=2}^{m} \prod_{L\in I(V)} \exp\{-\mathcal{S}^{\rm NLL}(V,L)\} \tag{13.3}$$

with

$$\mathcal{S}^{\rm NLL}(V,L) = \int_{d(V+1)}^{d(V)} \frac{d\lambda}{\lambda}\frac{\alpha_{\rm s}(\sqrt{\lambda s})}{2\pi}\Bigg\{C_{f(L)}\log\left(\frac{d(V_{\rm s}(L))}{\lambda}\right) - \gamma_{f(L)} + \tag{13.4}$$

$$+\,\theta(f(L)\in\{u,\bar{u},d,\dots\})\left[\frac{C_{\rm A}}{2} - C_{\rm F}\right]\log\left(\frac{d(V_{\rm s}(L))}{d(V_{\rm o}(L))}\right)\Bigg\}.$$

Here we interpret the lower limit $d(V+1)$ to be $d_{\rm ini}$ in the case $V = m$. This represents a simple algebraic reshuffling of the original expression that appears in eq. (13.1). What is more significant than the algebra is the interpretation. The total Sudakov factor is a product of factors, one for each splitting vertex V. Each of these factors represents the propagation of the system between scales $d(V)$ and $d(V+1)$. The factor for propagation from $d(V)$ and $d(V+1)$ is a product of factors, one for each parton line L that occurs in the intermediate state after splitting vertex V. Each of these factors, $\exp\{-\mathcal{S}(V,L)\}$, represents the probability that parton L in the intermediate state following vertex V did not split at a resolution parameter between $d(V)$ and $d(V+1)$. In the case of the last vertex, $V = m$, $\exp\{-\mathcal{S}(V,L)\}$ represents the probability that parton L in the intermediate state following vertex m did not split at a resolution parameter between $d(m)$ and $d_{\rm ini}$.

In the Sudakov factor, we integrate over the resolution parameter $\lambda = k_T^2/s$ of a virtual splitting. As we will see, we can understand the logarithms in the integrand as the result of integrating over the angle of a virtual splitting:

$$\begin{aligned}
\log\left(\frac{d(V_{\rm s}(L))}{\lambda}\right) &= \int_{\lambda\, s/E_{\rm s}^2}^{d(V_{\rm s}(L))\, s/E_{\rm s}^2}\frac{d\theta^2}{\theta^2}\,,\\
\log\left(\frac{d(V_{\rm s}(L))}{d(V_{\rm o}(L))}\right) &= \int_{d(V_{\rm o}(L))\, s/E_{\rm s}^2}^{d(V_{\rm s}(L))\, s/E_{\rm s}^2}\frac{d\theta^2}{\theta^2}\,,\\
\log\left(\frac{d(V_{\rm s}(L))}{\lambda}\right) - \log\left(\frac{d(V_{\rm s}(L))}{d(V_{\rm o}(L))}\right) &= \int_{\lambda\, s/E_{\rm s}^2}^{d(V_{\rm o}(L))\, s/E_{\rm s}^2}\frac{d\theta^2}{\theta^2}\,.
\end{aligned}\tag{13.5}$$

Here $E_{\rm s}$ is the energy of the least energetic of the two partons that were produced at vertex $V_{\rm s}(L)$.

The angle limits that appear in these formulas can be understood on the basis of "angular ordering" [16]. There are three types of splitting vertices to consider, $g \to gg$, $q \to qg$, and $g \to q\bar{q}$. Then the result in eq. (13.4) can be understood as the result of stringing together the three cases in all possible combinations.

The case of a $g \to gg$ vertex, V, is easiest. Typically, one of the daughter gluons is "hard" and moves in almost the direction of the mother gluon. The other gluon is softer and moves at an angle $\theta_V = k_{T,V}/E_V$ with respect to the mother direction, where E_V is the energy of the softer daughter gluon. That is,

$$\theta_V^2 = \frac{d(V)s}{E_V^2}\,. \tag{13.6}$$

Suppose that we now emit a soft gluon with energy E from the two daughter partons. The soft gluon has transverse momentum k_T with respect to the daughter parton from vertex

V. The resolution parameter for the soft gluon parameter is $\lambda = k_T^2/s$ and we integrate over λ in eq. (13.4). The emission angle θ of this soft gluon is given by $\theta^2 = \lambda\, s/E^2$. The smallest that θ^2 could be is $\theta^2_{\rm min} = \lambda\, s/E^2_{\rm max}$, where we can take $E_{\rm max}$ to be the energy of whichever parton is the source of the emission. Now suppose that the emission angle is less than the angle θ_V of the $g \to gg$ splitting. Then we get incoherent emission, with color factor $C_{\rm A}$, from each of the daughters. If we emit a soft gluon at a larger angle from the two daughters, the gluon does not resolve the daughters and instead sees the color of the mother. Thus large angle emission comes with a color factor $C_{\rm A}$. Thus we get soft gluon emission with a color factor $C_{\rm A}$ for either small or large angles with respect to the hard daughter direction, and we get *additional* radiation with a color factor $C_{\rm A}$ for small angles with respect to the direction of the soft daughter gluon.

For emission from the soft daughter, with a factor $C_{\rm A}$, we have an angular range $d(V)s/E_V^2 > \theta^2 > \lambda\, s/E_V^2$, where we have set $E_{\rm max} = E_V$. In this case, vertex V is the starting vertex for the soft daughter, so that the angular range is $d(s)s/E_s^2 > \theta^2 > \lambda\, s/E_s^2$, as in the first line of eq. (13.5).

For emission from the hard daughter plus coherent emission from both daughters, we have, to start with, a maximum angle equal to the angle of the vertex at which the mother gluon was a daughter gluon in a splitting. However, if the mother was the hard daughter parton in a $g \to gg$ splitting, we can also have coherent emission involving the mother's sister. Continuing in this way, we get emission with color factor $C_{\rm A}$ up to a maximum squared angle θ_s^2 where vertex s is the vertex at which the mother of the hard daughter started as a (relatively) soft daughter of energy E_s (either in a $g \to gg$ splitting or in a $q \to qg$ splitting). That is $\theta_s^2 = d(s)s/E_s^2$. The minimum angle is given by $\theta^2_{\rm min} = \lambda\, s/E^2_{\rm max}$ where we can approximate the hard daughter energy $E_{\rm max}$ by E_s. Thus the angular range for emissions from the hard daughter is $d(s)s/E_s^2 > \theta^2 > \lambda\, s/E_s^2$, as in the first line of eq. (13.5).

What about a $q \to qg$ vertex? Typically, the daughter quark is "hard" and moves in almost the direction of the mother quark. The daughter gluon is softer and moves at an angle k_T/E with respect to the mother direction. Suppose that we emit a soft gluon from the daughter partons with an emission angle less than the angle of the $q \to qg$ splitting. Then we get incoherent emission, with color factors $C_{\rm F}$ or $C_{\rm A}$ respectively, from the quark and gluon daughters. If we emit a soft gluon at a larger angle from the two daughters, the gluon does not resolve the daughters and instead sees the color of the mother. Thus large angle emission comes with a color factor $C_{\rm F}$. That is, we get soft gluon emission with a color factor $C_{\rm F}$ for either small *or* large angles, and we get additional radiation with a color factor $C_{\rm A}$ for small angles.

These two cases account for the factor $\log(d(V_{\rm s}(L))/\lambda)$ that appears in the first term in eq. (13.4). To understand the remaining term in eq. (13.4), we need to consider a $g \to q\bar{q}$ vertex.

In the case of a $g \to q\bar{q}$ vertex, both of the daughter partons are typically quite hard. We have incoherent soft gluon emission with color $C_{\rm F}$ from the two daughter partons as long as the emission angle is less than the splitting angle at the $g \to q\bar{q}$ vertex. For larger angles, we have soft gluon emission with color factor $C_{\rm A}$ because the soft gluon sees the net color of the $q\bar{q}$ pair, which is the color of the mother gluon. It is convenient to ascribe half of the large angle emission probability to each of the quark and the antiquark daughters. That gives an emission probability $C_{\rm A}/2$ for each. We can understand the Sudakov factor in eq. (13.4) in the case that $f(L) \in \{u, \bar{u}, d, \dots\}$ as describing an emission probability $C_{\rm F}$ for emissions from the quark line with angles smaller than the angle of the $q\bar{q}$ vertex at which the quark line was created and with a factor $C_{\rm A}$ for larger angles, up to the angle of the vertex $V_{\rm s}(L)$ at which the gluon that split to make the $q\bar{q}$ pair had its start.

13.2 Modified W for the Born cross section

Now that we understand the physical picture behind eqs. (13.3) and (13.4), we can propose a small modification. As was pointed out in [11] using rather different language, the term in eq. (13.4) proportional to $[C_{\rm A}/2 - C_{\rm F}]$ may seem unnecessarily complicated and could perhaps be eliminated. Note that this contribution arises only when there is a $g \to q\bar{q}$ vertex, whereas $g \to q\bar{q}$ splitting is rather unlikely, particularly because the $g \to gg$ comes with two logarithms while the $g \to q\bar{q}$ splitting comes with only one logarithm. Furthermore, the color factor is small compared to $C_{\rm F}$: $[C_{\rm A}/2 - C_{\rm F}]/C_{\rm F} = 1/(N_{\rm c}^2 - 1) = 1/8$. Shower Monte Carlo programs typically drop contributions that are of relative order $1/N_{\rm c}^2$, so it seems appropriate to do so here. For these reasons, our default choice is to drop this term and instead define $W_m(\{p,f\}_m)$ using

$$W_m(\{p,f\}_m) = \prod_{V>2} \frac{\alpha_{\rm s}\big(\sqrt{d(V)s}\big)}{\alpha_{\rm s}\big(\mu_{\rm R}\big)} \prod_{V=2}^{m} \prod_{L\in I(V)} \exp\{-\mathcal{S}(V,L)\}\,, \tag{13.7}$$

with (using $d(m+1) \equiv d_{\rm ini}$)

$$\mathcal{S}(V,L) = \int_{d(V+1)}^{d(V)} \frac{d\lambda}{\lambda} \frac{\alpha_{\rm s}(\sqrt{\lambda s})}{2\pi} \left\{ C_{f(L)} \log\!\left(\frac{d(V_{\rm s}(L))}{\lambda}\right) - \gamma_{f(L)} \right\}. \tag{13.8}$$

Since we include the factor W in the perturbative matrix element in $\sigma_m^{\rm B+S}$ and we wish to maintain the NLO accuracy of the calculation, we need to keep track of the first term in the perturbative expansion of W, eq. (8.3),

$$W_m(\{p,f\}_m) = 1 + \frac{\alpha_{\rm s}(\mu_{\rm R})}{2\pi}\, W_m^{(1)}(\{p,f\}_m) + \cdots. \tag{13.9}$$

Using eqs. (13.7) and (13.8) and using eq. (2.8) to express $\alpha_{\rm s}\big(\sqrt{d_V s}\big)$ in terms of $\alpha_{\rm s}\big(\mu_{\rm R}\big)$, we have

$$\begin{aligned} W_m^{(1)}(\{p,f\}_m) = &- \sum_{V=2}^{m} \sum_{L\in I(V)} \int_{d(V+1)}^{d(V)} \frac{d\lambda}{\lambda} \left\{ C_{f(L)} \log\left(\frac{d(V_{\rm s}(L))}{\lambda}\right) - \gamma_{f(L)} \right\} - \\ &- \sum_{V=3}^{m} \beta_0 \log\left(\frac{d(V)s}{\mu_{\rm R}^2}\right), \end{aligned} \tag{13.10}$$

where, again, $d(m+1) \equiv d_{\rm ini}$.

13.3 Sudakov factors in dipole splitting

Let us now look at the Sudakov exponential that occurs in the function $\boldsymbol{E}_{l,k}$, eq. (8.6). We set the evolution variable to be k_T^2/s, as in eq. (8.8). Here $k_T^2 = s_l y z(1-z)$ and k_T is also the argument of the running coupling according to eq. (8.9). Then the Sudakov exponential in eq. (8.6) is independent of l and k and is

$$\begin{aligned}\Delta(r) = \prod_{l'} \exp\biggl(- \int_r^\infty dr' \int_0^1 \frac{dy'}{y'} \int_0^1 dz'\, \delta(r' - s_{l'} y' z'(1-z')) \times \\ \times\, \theta\left(\frac{r'}{\max\{z'^2, (1-z')^2\}} < d_{\rm ini} s \right) \times \\ \times\, \frac{\alpha_s(\sqrt{r'})}{2\pi} \left\langle \boldsymbol{S}(y', z', f_{l'}) \right\rangle \biggr) . \end{aligned} \tag{13.11}$$

Here the product runs over all final state partons l'. After using eqs. (11.11) and (11.12) for the splitting functions, we can perform the $z'$and y' integrals in the exponent to obtain

$$\begin{aligned}\Delta(r) = \prod_{l'} \exp\biggl(- \int_{r/s}^{d_{\rm ini}} \frac{d\lambda}{\lambda} \frac{\alpha_{\rm s}(\sqrt{\lambda s})}{2\pi} \biggl[C_{f(l')} \log\left(\frac{s_{l'}/s}{\lambda}\right) - \gamma_{f(l')} + \\ + \mathcal{O}\left(h\left(\frac{d_{\rm ini}}{\lambda}\right)\right) + \mathcal{O}\left(\frac{\lambda}{s_{l'}/s} \log\left(\frac{s_{l'}/s}{\lambda}\right)\right)\biggr]\biggr) . \end{aligned} \tag{13.12}$$

Here $s_{l'}/s$ lies between $d_{\rm ini}$ and 1. This form for the exponent applies when $\lambda \ll s_{l'}/s$, with a correction that vanishes when $\lambda/(s_{l'}/s) \to 0$. There is a second correction term that is bounded by a constant times

$$h\left(\frac{d_{\rm ini}}{\lambda}\right) = \theta\left(\lambda > \frac{d_{\rm ini}}{4}\right) \log\left(\frac{1}{2[1 - \sqrt{\lambda/d_{\rm ini}}]}\right) . \tag{13.13}$$

This correction comes from the difference between having a simple cut $r'/s < d_{\rm ini}$ and having our more complicated cut $r'/s < d_{\rm ini} \times \max\{z'^2, (1-z')^2\}$. The two correction terms make finite contributions to the Sudakov exponent when $r/s \ll s_{l'}/s$ with $d_{\rm ini}$ either small compared to $s_{l'}/s$ or of order $s_{l'}/s$ (but in any case bigger than r/s).

We see that, to the accuracy of dropping non-logarithmic terms in the exponent, we have

$$\Delta(r) \approx \prod_l \exp\left[- \int_r^{d_{\rm ini}} \frac{d\lambda}{\lambda} \frac{\alpha_{\rm s}(\sqrt{\lambda s})}{2\pi} \left\{ C_{f(l)} \log\left(\frac{s_l/s}{\lambda}\right) - \gamma_{f(l)} \right\}\right] . \tag{13.14}$$

Compare this to the factor in eq. (13.7), corresponding to the probability that the final state particles not split between the scale $d(m)$ and the limiting scale $d_{\rm ini}$. According to eq. (13.8), this is

$$\prod_l e^{-\mathcal{S}(m,l)} = \prod_l \exp\left[- \int_{d_{\rm ini}}^{d(m)} \frac{d\lambda}{\lambda} \frac{\alpha_{\rm s}(\sqrt{\lambda s})}{2\pi} \left\{ C_{f(l)} \log\left(\frac{d(V_{\rm s}(l))}{\lambda}\right) - \gamma_{f(l)} \right\}\right] . \tag{13.15}$$

Let us choose

$$s_l = s\, d_{V(l)} \, . \tag{13.16}$$

Then

$$\Delta(r)\prod_l e^{-\mathcal{S}(m,l)} = \prod_l \exp\left[-\int_r^{d(m)}\frac{d\lambda}{\lambda}\frac{\alpha_s(\sqrt{\lambda s})}{2\pi}\left\{C_{f(l)}\,\log\!\left(\frac{d(V_s(l))}{\lambda}\right)-\gamma_{f(l)}\right\}\right], \tag{13.17}$$

corresponding to the final state particles not splitting between the scale $d(m)$ and the scale r, at which one of them does split. There is no dependence on $d_{\rm ini}$ in this product.

We thus see the purpose of including the factor $W_m(\{p,f\}_m)$ in the formula for $\sigma_m^{\rm B+S}$: without it, there would be a dependence on $d_{\rm ini}$ from the Sudakov factors for showering below the resolution scale $d_{\rm ini}$. More precisely, this is the purpose of including the factor in $W_m(\{p,f\}_m)$ relating to propagation from scale $d(m)$ to $d_{\rm ini}$. The other Sudakov factors in $W_m(\{p,f\}_m)$ are equally important. The complete expression for $\sigma_m^{\rm B+S}$ has Sudakov factors from three sources. Evolution from d_2 to d_m is included in $W_m(\{p,f\}_m)$. Evolution from d_m to $r = d_{m+1}$ is included partly in $W_m(\{p,f\}_m)$ and partly in the Sudakov factor $\Delta(r)$ that is part of $\boldsymbol{E}_{l,k}$, as we have just seen. Then there is a succession of shower splitting scales that we have represented as being part of $I(\{p,f\}_m; l,k,Y_l)$. This showering will involve Sudakov factors for evolution from $m+1$ to $m+2$, from $m+2$ to $m+3$, and so forth. If we hold all of the parton momenta fixed and change $d_{\rm ini}$ from just less than d_m to just greater than d_m, then the same event history counts not as part of $\sigma_m^{\rm B+S}$ but rather as part of $\sigma_{m+1}^{\rm B+S}$. The integrand representing the probability for this event history in $\sigma_m^{\rm B+S}$ will be somewhat different from the integrand for the same event history in $\sigma_{m+1}^{\rm B+S}$, but the two integrands will be approximately the same. The main points are that the Sudakov factors from all three sources are approximately the same and that the $m+1$-parton squared matrix element is approximately the m-parton squared matrix element times the dipole splitting functions.

13.4 Factors W for real and virtual corrections

We have defined the factor $W_m(\{p,f\}_m)$ in $\sigma_m^{\rm B+S}$ and examined its structure and how it meshes with the Sudakov factors in the splitting function $\boldsymbol{E}_{l,k}$. We now turn to the function $W_m^{\rm V+S}(\{p,f\}_m)$ that appears in $\sigma_m^{\rm V+S}$, eq. (8.47), and the function $W_{m+1}^{\rm R+S}(\{p,f\}_{m+1})$ that appears in $\sigma_m^{\rm R+S}$, eq. (8.43).

For $W_m^{\rm V+S}(\{p,f\}_m)$, we have an m-parton final state that is constructed to be resolvable at scale $d_{\rm ini}$. It thus seems sensible to use the same reweighting function that we used for the Born contribution,

$$W_m^{\rm V+S}(\{p,f\}_m) = W_m(\{p,f\}_m)\,. \tag{13.18}$$

It is worth noting that this choice is not as obvious as it was in the for $\sigma_m^{\rm B+S}$. Recall that $\sigma_m^{\rm V+S}$ is based on graphs with a virtual loop (together with a subtraction that keeps the virtual partons from being very soft or collinear with the external particles). If the virtual partons always had momenta of order $\sqrt{s}$ then the choice (13.18) would be physically well motivated. However, the momentum scales in the virtual loop can be intermediate between $\sqrt{s}$ and the scales of the parton splittings in the synthetic shower history constructed from the final state $\{p,f\}_m$. For this reason, it could well be that the "best" choice for $W_m^{\rm V+S}(\{p,f\}_m)$ is something more subtle than that given in eq. (13.18).

For $W_m^{\rm R+S}(\{p,f\}_{m+1})$, we have an $m+1$-parton final state, so that we have resolution scales $d(1) \geq \cdots \geq d(m) \geq d(m+1)$ in the synthetic shower graph generated from the final state $\{p,f\}_{m+1}$. In the main term of $\sigma_m^{\rm R+S}$, we have $d(m) > d_{\rm ini} > d(m+1)$. There are then subtraction terms that remove the leading singularity when $d(m) \gg d(m+1)$. For this reason, $d(m+1)$ is typically not much smaller than $d_{\rm ini}$, nor is $d(m)$ much larger. We therefore define $W_m^{\rm R+S}(\{p,f\}_{m+1})$ to include a reweighting factor for α_s at each strong interaction vertex and a Sudakov factor giving the probability that the partons did not radiate between each pair of vertices:

$$W_m^{\rm R+S}(\{p,f\}_{m+1}) = \prod_{V=3}^{m+1} \frac{\alpha_{\rm s}\big(\sqrt{d(V)s}\big)}{\alpha_{\rm s}\big(\mu_{\rm R}\big)} \prod_{V=2}^{m} \prod_{L\in I(V)} \exp\{-\mathcal{S}(V,L)\} \tag{13.19}$$

with $\mathcal{S}(V,L)$ as defined in eq. (13.8). The last Sudakov factor takes us from vertex m, after which there are m partons, to vertex $m+1$ at which one of these partons splits with a scale $d(m+1)$ to make an $m+1$-parton state. Subsequent evolution, as given in the Monte Carlo interface function $\tilde{I}^{\rm R+S}(\{p,f\}_{m+1})$ in eq. (8.43), then starts at splitting scale $d(m+1)$. Again, it is worth noting that this choice is not as obvious as it was in the for $\sigma_m^{\rm B+S}$ because of the subtraction term in $\sigma_m^{\rm R+S}$. It could well be that the "best" choice for $W_m^{\rm R+S}(\{p,f\}_m)$ is something more subtle than that given in eq. (13.19).

An alternative is to take

$$W_m^{\rm V+S}(\{p,f\}_m) = W_m^{\rm R+S}(\{p,f\}_{m+1}) = 1\,. \tag{13.20}$$

This choice removes some sensible physics built into $W_m^{\rm V+S}$ and $W_m^{\rm R+S}$. However, it does have a technically useful feature. Suppose that we take a measurement function $F = 1$, so that $\sigma_m^{\rm B+S} + \sigma_m^{\rm V+S} + \sigma_m^{\rm R+S}$ is the contribution to the total cross section from final states with precisely m jets. The Born contribution, $\sigma_m^{\rm B+S}$, contains terms of all orders in $\alpha_{\rm s}$ starting at order $\alpha_{\rm s}^{B_m}$. It does not have the complete contribution at order $\alpha_{\rm s}^{B_m+1}$ and beyond, but it does have the correct leading and next-to-leading logarithms of $d_{\rm ini}$ [11] (at leading order in $1/N_{\rm c}^2$, since we have modified W_m a little). The addition of $\sigma_m^{\rm V+S}$ and $\sigma_m^{\rm R+S}$ fills in the missing pieces at order $\alpha_{\rm s}^{B_m+1}$. It also can add contributions at higher orders. However, if we choose $W_m^{\rm V+S}(\{p,f\}_m) = W_m^{\rm R+S}(\{p,f\}_{m+1}) = 1$, then $\sigma_m^{\rm V+S}$ and $\sigma_m^{\rm R+S}$ are exactly proportional to $\alpha_{\rm s}^{B_m+1}$, with no contributions at higher orders. They provide just what is needed at order $\alpha_{\rm s}^{B_m+1}$ and no more. To see this, note that in the expression (8.43) for $\sigma_m^{\rm R+S}$, the factor $\tilde{I}$ is 1 when $F = 1$ because of the property (8.18) of $\tilde{I}$. When we also set $W_m^{\rm R+S}(\{p,f\}_m)$ to 1, we get an expression proportional to $\alpha_{\rm s}^{B_m+1}$ with no further dependence on $\alpha_{\rm s}$. Similarly, the choice $W_m^{\rm V+S}(\{p,f\}_m) = 1$ makes $\sigma_m^{\rm V+S}$, eq. (8.47), exactly proportional to $\alpha_{\rm s}^{B_m+1}$.

14. What size is the resolution parameter?

The algorithm described in this paper depends on a jet resolution parameter $d_{\rm ini}$, which plays a role similar to that of the factorization scale in calculations of cross sections for hard processes with one or two hadrons in the initial state. We anticipate that $d_{\rm ini}$ would be a parameter in a computer code that implements this algorithm, so that a user could choose its value. In this section, we describe some considerations that would go into making the choice.

Suppose that we calculate $\sigma[F]$ for an infrared safe N-jet observable that measures large momentum scale features of events and hence does not involve large logarithms. An example for $N = 3$ is $F = T_4$, the fourth moment of the thrust distribution, eq. (1.1). More generally, we can consider the nth moment of the trust distribution for $n \geq 2$,

$$\sigma[T_n] = \int_0^1 dt\,(1-t)^n\,\frac{d\sigma}{dt}\,. \tag{14.1}$$

Given that we want to calculate $\sigma[F]$ for an observable in this class, we seek to choose the resolution scale $d_{\rm ini}$ so the dominant contribution to $\sigma[F]$ comes from $\sigma_m[F]$, with $m = N$. Let us examine why one would like $\sigma_N[F]$ to be dominant and how dominant it should be.

We first consider the comparison of $\sigma_N[F]$ to $\sigma_{N+1}[F]$, supposing that $\sigma_m[F]$ for $m \leq N-1$ is negligible. We note that the cross section $\sigma[F] = \sum_m \sigma_m[F]$ has a perturbative expansion that starts at order $\alpha_{\rm s}^{B_N}$ and has higher order contributions. For a next-to-leading order calculation, we need the next contribution, of order $\alpha_{\rm s}^{B_N+1}$. The partial cross section $\sigma_N[F]$ also has a perturbative expansion that starts at order $\alpha_{\rm s}^{B_N}$. As we have seen, the algorithm presented in this paper reproduces the order $\alpha_{\rm s}^{B_N}$ and $\alpha_{\rm s}^{B_N+1}$ terms of this expansion. The perturbative expansion for $\sigma_{N+1}[F]$ begins at order $\alpha_{\rm s}^{B_N+1}$. Thus part of the order $\alpha_{\rm s}^{B_N+1}$ correction to $\sigma[F]$ is in $\sigma_N^{\rm R+S}[F]$ and $\sigma_N^{\rm V+S}[F]$ and part is in $\sigma_{N+1}^{\rm B+S}[F]$. (Part is in $\sigma_m[F]$ for $m \leq N-1$, but we suppose for the moment that this part is negligible.)

Consistently with this counting of powers of $\alpha_{\rm s}$, we expect $\sigma_{N+1}[F] \sim \alpha_{\rm s}\,\sigma_N[F]$. However, suppose that we were to choose such a small value for $d_{\rm ini}$ that $\alpha_{\rm s}\log^2(1/d_{\rm ini})$ is of order 1. Then the cross section $\sigma_N[F]$ that we hoped was dominant would be suppressed and the cross section would be shifted to $\sigma_m[F]$ with $m = N+1, N+2, \dots$. This would not much matter at leading order since the k_T-jet matching scheme arranges that the sum of the $\sigma_m^{\rm B+S}[F]$ is very accurately independent of $d_{\rm ini}$. However, it is possible that the perturbative correction terms of order $\alpha_{\rm s}^{B_N+1}$ can be lost among terms like $\alpha_{\rm s}^{B_N+1}\times\alpha_{\rm s}\log^2(1/d_{\rm ini})$. Such a term is beyond the next-to-leading perturbative order for $\sigma_N[F]$ and beyond the next-to-leading logarithm approximation also. Terms like this are always present in $\sigma_{N+1}^{\rm B+S}[F]$, $\sigma_N^{\rm R+S}[F]$ and $\sigma_N^{\rm V+S}[F]$, but they are numerically important only when $d_{\rm ini}$ is very small. To ensure that these contributions are not numerically important, we can ask that the relation between $\sigma_N[F]$ and $\sigma_{N+1}[F]$ follow the perturbative expectation,

$$\sigma_{N+1}[F] \lesssim \alpha_{\rm s}\,\sigma_N[F]\,. \tag{14.2}$$

With a computer implementation of the algorithm described here at hand, one can also check that $\sigma_N^{\rm R+S}[F]$ and $\sigma_N^{\rm V+S}[F]$ are of the right perturbative size (that is, $\alpha_{\rm s}\,\sigma_N[F]$) and that the complete $\sigma[F]$ is accurately reproducing the purely perturbative result for the same quantity, which can be obtained by simply turning off the shower part of the program.

Consider next the partial cross section $\sigma_{N-1}[F]$, with $N-1$ resolvable jets. The perturbative expansion for $\sigma_{N-1}[F]$ has the form (1.4)

$$\sigma_{N-1}[F] = C_{N-1,0}[F]\,\alpha_{\mathrm{s}}^{B_N-1}(Q) + C_{N-1,1}[F]\,\alpha_{\mathrm{s}}^{B_N}(Q) + \cdots\,. \tag{14.3}$$

This compares to the expansion for $\sigma_N[F]$,

$$\sigma_N[F] = C_{N,0}[F]\,\alpha_{\mathrm{s}}^{B_N}(Q) + C_{N,1}[F]\,\alpha_{\mathrm{s}}^{B_N+1}(Q) + \cdots\,. \tag{14.4}$$

The first perturbative coefficient for $\sigma_{N-1}[F]$ vanishes since, by assumption, $F(f) = 0$ for a state with $N-1$ partons. Fortunately, we have the next term,[6] proportional to $\alpha_{\mathrm{s}}^{B_N}$. We do not have the complete coefficient proportional to $\alpha_{\mathrm{s}}^{B_N+1}$. Thus it seems that we do not have the $\alpha_{\mathrm{s}}^{B_N+1}$ accuracy that we want. However, $F(f) = 0$ for a state with $N-1$ very narrow jets. Furthermore $\sigma_{N-1}[F]$ is the contribution from final states with $N-1$ jets that cannot be further resolved at scale d_{ini}. If we take d_{ini} to be small, the $N-1$ jets are guaranteed to be narrow. We conclude that all of the coefficients $C_{N-1,j}[F]$ will be small if we choose d_{ini} to be small. Since we are missing one factor of α_{s} in the expansion of $\sigma_{N-1}[F]$, we can simply demand that d_{ini} be chosen so that the coefficients $C_{N-1,j}[F]$ are smaller than the corresponding coefficients $C_{N,j-1}[F]$ in $\sigma_N[F]$ by a numerical factor that is approximately equal to α_{s} or else smaller. That is, we ask that

$$\sigma_{N-1}[F] \lesssim \alpha_{\mathrm{s}}\,\sigma_N[F]\,. \tag{14.5}$$

Can one really satisfy eqs. (14.2) and (14.5)? Consider the three-jet observable T_4, eq. (14.1). Using Pythia with $\sqrt{s} = M_Z$ to estimate $\sigma_m[T_4]$ we find that $\sigma_3[T_4]$ is largest near $d_{\mathrm{ini}} = 0.04$. At this value of d_{ini}, we have $\sigma_4[T_4] = 0.08\,\sigma_3[T_4]$ and $\sigma_2[T_4] = 0.12\,\sigma_3[T_4]$, thus satisfying eqs. (14.2) and (14.5).

The case of the second moment of the thrust distribution, $F = T_2$, is instructive. Again taking $\sqrt{s} = M_Z$, we find that $\sigma_3[T_2]$ is largest near $d_{\mathrm{ini}} = 0.02$. At this value of d_{ini}, we have $\sigma_4[T_2] = 0.16\,\sigma_3[T_2]$ and $\sigma_2[T_2] = 0.35\,\sigma_3[T_2]$. In this case the two jet contribution is not as suppressed as one might like. We do not interpret this as indicating a difficulty with the k_T-jet matching scheme. Rather we take it as an indication that, after accounting for showering and hadronization, $\sigma[T_2]$ gets a 30% contribution from the two jet region (defined with jet resolution 0.02) even though T_2 is nominally an infrared-safe three-jet quantity. For this reason, a pure NLO fixed order calculation is not as reliable a calculation as one might like and one should not be concerned if the algorithm in this paper based on k_T-jet matching does not exactly match the pure NLO calculation. To the extent that they turn out to differ, we expect that the k_T-jet matching calculation with some perturbative accuracy sacrificed in favor of showers and hadronization is the better calculation.

These examples suggest that a choice of d_{ini} in the range $\alpha_{\mathrm{s}}^2 < d_{\mathrm{ini}} < \alpha_{\mathrm{s}}$ might be appropriate in practical cases. We can also ask whether a choice of d_{ini} in the range $\alpha_{\mathrm{s}}^2 < d_{\mathrm{ini}} < \alpha_{\mathrm{s}}$ would work in the asymptotic limit $\alpha_{\mathrm{s}} \to 0$.

[6]Since the required NLO calculations are available for N jets, we can assume that they are available for $N-1$ jets.

Consider first the condition in eq. (14.5). It is pretty straightforward to see that for an infrared safe N-jet observable F, $\sigma_{N-1}[F]$ will vanish like a power of $d_{\rm ini}$ (times logarithms) for $d_{\rm ini} \to 0$. For example, for the nth moment of the thrust distribution, eq. (14.1), one finds a limiting form $\sigma_2[T_n] \propto d_{\rm ini}^{n/2}$. Thus if we take $d_{\rm ini} \sim \alpha_{\rm s}$ or smaller, eq. (14.5) will be satisfied asymptotically as long as $n \geq 2$. Following the philosophy stated above for $\sigma_2[T_2]$ when $\alpha_{\rm s} = \alpha_{\rm s}(M_Z)$, we might take the observables T_n for $n < 2$ to be sufficiently contaminated by two jet physics that we would not need to satisfy eq. (14.5) for those observables.

Consider next the condition in eq. (14.2). Since the perturbative expansion of $\sigma_{N+1}[F]$ begins with one more power of $\alpha_{\rm s}$ than the perturbative expansion of $\sigma_N[F]$, eq. (14.2) appears to follow. There is a potential problem from the Sudakov exponentials, which could suppress $\sigma_N[F]$ relative to $\sigma_{N+1}[F]$. The exponents contain factors $\alpha_{\rm s}\,\log^2(1/d_{\rm ini})$. With $d_{\rm ini} \sim \alpha_{\rm s}$ or $d_{\rm ini} \sim \alpha_{\rm s}^2$, the Sudakov exponent becomes $const.\,\alpha_{\rm s}\,\log^2(1/\alpha_{\rm s})$. In the limit $\alpha_{\rm s} \to 0$, these factors get smaller. Thus the Sudakov exponentials do not change the perturbative power counting and eq. (14.2) is satisfied.

Thus both numerical examples based on Pythia at $\alpha_{\rm s} = \alpha_{\rm s}(M_Z)$ and an analysis of the limit $\alpha_{\rm s} \to 0$ suggest that a choice of $d_{\rm ini}$ in the range $\alpha_{\rm s}^2 < d_{\rm ini} < \alpha_{\rm s}$ might be useful. When one has a working program, one will want to revisit these questions with a numerical analysis using the program.

15. The Monte Carlo interface function

In section 8.5, we introduced a function $I(\{p,f\}_m; l, k, Y_l)$ that represents the expectation value of the observable in hadronic states that are generated by a shower Monte Carlo event generator beginning from initial conditions represented by the partonic state $\{p,f\}_m$ and the information about the splitting of parton l contained in the variables l, k, Y_l. This function is used for $\sigma_m^{\rm B+S}$. We also introduced a similar function $\tilde{I}(\{p,f\}_n)$ that is used for $\sigma_m^{\rm R+S}$ and $\sigma_m^{\rm V+S}$ (and may optionally be used for $\sigma_m^{\rm B+S}$ for $m > m_{\rm NLO}$). For the matching to a NLO calculation, we needed only certain basic properties of these functions, given in eqs. (8.16) and (8.17), and for $\tilde{I}$, in eq. (8.19).

In this section, we delve a little further into the requirements for the shower Monte Carlo event generator. There are quite a number of successful shower algorithms available and it is beyond our scope to specify any algorithm in detail. However it may be useful to say a little about the initial conditions for the shower Monte Carlo program. In particular, we note that we are handing the Monte Carlo a partially developed shower and that the Monte Carlo needs to begin at the point where the previous shower simulation left off.

Consider, then, the function $I(\{p,f\}_m; l, k, Y_l)$ used for $\sigma_m^{\rm B+S}$. There is an m-parton state that can be thought of as having been created by a showering process with a hardness cutoff $d_{\rm ini} < d_m(\{p,f\}_m$. After that, one of the partons (labelled l) splits. Suppose for the sake of concreteness that this dipole splitting is based on the evolution variable of eq. (8.8),

$$r = s_l\, yz(1-z)\,. \tag{15.1}$$

Let us define a transverse momentum $k_\perp$ for this splitting according to eq. (10.9). With this choice, we have

$$|k_\perp^2| = 2p_l\cdot p_k\, yz(1-z) = \frac{2p_l\cdot p_k}{s_l}\, r\,. \tag{15.2}$$

Note that for a given value of the evolution variable r, $|k_\perp^2|$ for the splitting of parton l with the participation of spectator parton k depends on l and k.

Now, all of the partons are allowed to split, and the one that does is parton l. The others did not split at an evolution variable above the value r. That is, parton l', with the aid of spectator k', did not split with $r' > r$. Further evolution of these partons should then be restricted to the range $r' < r$. That is

$$|k_\perp'^2| < \frac{2p_{l'}\cdot p_{k'}}{s_{l'}}\frac{s_l}{2p_l\cdot p_k}|k_\perp^2|\,. \tag{15.3}$$

A restriction like this can be imposed in the chosen shower Monte Carlo program by using a veto algorithm, as described for instance in ref. [11]. The only problem may be that the Monte Carlo program may know the index l' of the parton that it is proposing to split, but may not have a choice for a corresponding spectator parton k'. A sensible choice would be to let k' be one of the final state partons to which parton l' is color connected (at leading order in $1/N_c$) according to a rule based on the synthetic splitting diagram obtained by applying the k_T-jet algorithm to the m-parton state and adding the one splitting of parton l. (For a gluon l' there are two such color connected partons and one would choose either of them with probability 1/2.) An alternative that avoids selecting a spectator parton is to impose

$$|k_\perp'^2| < \max_{k'}\left\{\frac{2p_{l'}\cdot p_{k'}}{s_{l'}}\frac{s_l}{2p_l\cdot p_k}|k_\perp^2|\right\}. \tag{15.4}$$

For the splitting of one of the daughters of parton l, one may simply impose

$$|k_\perp'^2| < |k_\perp^2|\,. \tag{15.5}$$

Of course, if the splitting angle is not the evolution variable used in the shower Monte Carlo program used for secondary showers, then the program should separately arrange for angular ordering.

Consider next the function $\tilde I(\{p,f\}_m)$ as used for $\sigma_m^{\rm V+S}$. The final state, which is the initial state for the Monte Carlo program, consists of m partons that can be thought of as having been created by a showering process, with a hardness cutoff $d_{\rm ini} < d_m(\{p,f\}_m$. There is no further splitting before this state is passed to the Monte Carlo program. Therefore further splittings $l \to i + j$ should be generated with a cut $d(p_i,p_j) < d_{\rm ini}$. The same reasoning applies in the case that one elects to use $\tilde I$ without a dipole splitting for $\sigma_m^{\rm B+S}$ in the case $m > m_{\rm NLO}$, as described in section 9.

The case of $\tilde I(\{p,f\}_{m+1})$ as used for $\sigma_m^{\rm R+S}$ is similar. Here there are $m+1$ partons with $d_{m+1}(\{p,f\}_{m+1}) < d_{\rm ini}$. Thus it is $d_{m+1}(\{p,f\}_{m+1})$ rather than $d_{\rm ini}$ that should serve as the upper cutoff for splittings included in $\tilde I(\{p,f\}_{m+1})$.

16. Conclusion

We have proposed an algorithm for adding showers to next-to-leading order calculations for $e^+ + e^- \to N$ *jets*. This algorithm incorporates two kinds of matching, which we might call PS/NLO matching and m-jet/$(m+1)$-jet matching.

For the m-jet/$(m+1)$-jet matching, we make use of the k_T-jet matching scheme of Catani, Krauss, Kuhn, and Webber [11]. We divide the cross section $\sigma[F]$ corresponding to an observable F into contributions $\sigma_m[F]$ corresponding to final states with m hard jets. Here "hard" is defined using a jet resolution parameter $d_{\rm ini}$. In the case that F is an N-jet observable that is infrared-safe in the technical sense and is in fact insensitive to physics at momentum scales much below $\sqrt{s}$, we want the dominant contribution to $\sigma[F]$ to come from $\sigma_m[F]$ with $m = N$.

For PS/NLO matching, we are concerned with the m partons that emerge from a Born graph that contributes to $\sigma_m[F]$. One of these partons can split as the first step of a parton shower based on this graph. On the other hand, there are perturbative corrections to the same graph that also involve real or virtual splittings of the same m partons. The MC/NLO matching is needed in order to incorporate the parton shower and the NLO corrections without double counting. With the parton shower in place, we not only get $\sigma_m[F]$ correct to NLO, but we also get sensible answers to more detailed questions about the low momentum scale structure of the m jets that are used to calculate $\sigma_m[F]$. A pure NLO calculation cannot (sensibly) answer such questions.

We thus have a good description of hard physics and soft physics and even both at once. What about physics at an intermediate momentum scale? Suppose that F is such that $\sigma[F]$ is sensitive to physics at a scale K^2 that is similar to $d_{\rm ini}s$. An example would be the thrust distribution $d\sigma/dt$ with $K^2 \equiv (1-t)s$ similar to $d_{\rm ini}s$. In this case neither $\sigma_2[F]$ nor $\sigma_3[F]$ dominates the calculation of $\sigma[F]$. If we hold K^2 fixed and increase $d_{\rm ini}$ a little, $\sigma_2[F]$ decreases and $\sigma_3[F]$ decreases. If we use the Born level contributions $\sigma_m^{\rm B+S}[F]$, the approximations used in $\sigma_2^{\rm B+S}[F]$ and $\sigma_3^{\rm B+S}[F]$ are so similar that the sum $\sigma_2^{\rm B+S}[F] + \sigma_3^{\rm B+S}[F]$ is very insensitive to $d_{\rm ini}$. If we now include $\sigma_m^{\rm R+S}[F] + \sigma_m^{\rm V+S}[F]$, we improve the approximations by adding NLO correction terms. However, the correction terms $\sigma_2^{\rm R+S}[F] + \sigma_2^{\rm V+S}[F]$ and $\sigma_3^{\rm R+S}[F] + \sigma_3^{\rm V+S}[F]$ are not matched in the same sense that the leading terms $\sigma_2^{\rm B+S}[F]$ and $\sigma_3^{\rm B+S}[F]$ are matched. To do better at matching the correction terms for different numbers of jets would require next-to-leading order splitting kernels for the showers, which we do not have. For this reason, one may expect that adding the correction terms $\sigma_m^{\rm R+S}[F] + \sigma_m^{\rm V+S}[F]$ introduces some $d_{\rm ini}$ dependence into the result. Fortunately, the NLO correction terms are small, so the added $d_{\rm ini}$ dependence must also be small.

We have been quite specific about the subtraction scheme to use in the base NLO calculation. We have been deliberately less specific about the scheme for showering. The hardest splitting of a parton from a Born graph is to be done using the dipole splitting functions. Softer splittings can be handled by the reader's favorite shower Monte Carlo method as long as the initial conditions for this secondary showering are compatible with the organization of the part of the showering already completed.

It may be useful to compare the algorithm discussed in this paper with those in other papers. We can compare treatments of the Sudakov factor $\boldsymbol{E}_{l,k}$ in eq. (8.1), the reweighting factor W, and the factor I that represents further showering.

Consider first the exponent in the Sudakov factor $\boldsymbol{E}_{l,k}$ for the splitting that is to be matched to NLO perturbation theory, eq. (8.6). In the work of Frixione, Nason, and Webber [5, 5], this exponent is the splitting function of HERWIG. Thus one must subtract something from the NLO calculation that is derived from the order $\alpha_{\rm s}$ expansion of a HERWIG shower. As discussed by Nason [15], this means that the NLO calculation is to be tuned to the shower Monte Carlo. Additionally, since the HERWIG shower does not precisely match the perturbative probability for soft wide-angle gluon emission, one must also provide a special treatment for the soft region. The algorithm of Ref [8, 9] is based on the simple idea that, in a physical gauge, the cut two point function represents parton splitting, while the virtual two point function represents the probability not to split. For this reason, the Sudakov exponent is chosen as the integrand of the virtual two point function in the Coulomb gauge. There is a separate Sudakov exponential for emission of soft wide-angle gluons. This method works, but is not easily adapted to the dipole style of NLO calculations that most people use. In the present work, as in the leading order program of Sjöstrand and Skands [14], the hardest splitting in a shower gets a special treatment. Nason [15] has proposed a scheme of this style that is adapted to an angle ordered shower, as in HERWIG. In this scheme, the Sudakov exponent for the hardest splitting is the ratio of the exact matrix element $\langle\mathcal{M}(\{\hat{p},\hat{f}\}_{m+1})|\mathcal{M}(\{\hat{p},\hat{f}\}_{m+1})\rangle$ for $m+1$ partons to the matrix element $\langle\mathcal{M}(\{p,f\}_{m})|\mathcal{M}(\{p,f\}_{m})\rangle$ for m partons, before one of the partons is imagined to have split. The $m+1$-particle matrix element is to be divided into a sum of terms associated with individual parton splittings using an algorithm that is not completely specified in [15]. There is a certain conceptual simplicity to this choice, but it puts a very complicated function into the Sudakov exponent. In this paper, the Sudakov exponent is the angular average of the splitting function in the dipole subtraction scheme that is widely used for NLO calculations. With this choice, the Sudakov exponent is quite simple and at the same time it incorporates soft wide-angle gluon emissions exactly.

Consider next the reweighting factor W, whose main purpose is to provide a Sudakov suppression factor multiplying the matrix element according to a synthetic shower that corresponds to the Born level final state $\{p,f\}_m$. This factor is not present in either of the existing programs [5, 6] and [8 − 10] that add parton showers to an NLO calculation. The reweighting factor W is simply taken from the paper of Catani, Krauss, Kuhn and Webber [11] and adapted to the present case of an NLO calculation. There may be improvements possible. For instance, Lönnblad [17] and Mrenna and Richardson [18] have provided more sophisticated algorithms for calculating a function that plays this role but more closely matches the Sudakov factor that would actually be generated in a shower Monte Carlo program.

Consider finally the factor I that represents the expectation value of the observable after secondary showering and hadronization that is initiated from the partons produced at the Born level together with one hardest splitting. We have left this function as something that can be freely chosen subject only to a few very general conditions. If one wants a HERWIG style angle-ordered shower, the vetoed shower treatment of Catani, Krauss, Kuhn and Webber [11] is useful. Additionally, the recent paper of Nason [15] explains in some detail how an angle ordered-shower can be initiated after a first step consisting of the hardest splitting rather than the widest angle splitting. If one would like a P_T ordered shower that properly incorporates coherence, the recent paper of Sjöstrand and Skands [14] describes such a treatment, including the feature that emissions are generated in the order of decreasing overall P_T in the whole event rather than P_T along a single line. If one would like a shower based on the Color-Dipole Cascade Model of ARIADNE, then the work of Lönnblad [17] is to be recommended. We note that the most natural choice of shower to follow the initial splitting based on the Catani-Seymour dipole splitting functions and kinematic definitions would be a shower in which *all* of the splittings are based on this choice. A construction based on this choice is, however, beyond the scope of this paper. Instead, we leave the choice of secondary showering scheme open. We emphasize that the same NLO calculation with the hardest splitting included could then be coupled to shower Monte Carlo programs using a variety of splitting styles as long as the subsequent shower is initiated so as to account for the splitting that has already occurred.

The algorithms covered here include final state showers, but do not allow for strongly interacting partons in the initial state. The dipole subtraction scheme of ref. [12] covers initial state partons as well, but we thought it best to leave the case of initial state showers for a separate publication.

We certainly expect that the algorithm presented here can be improved. For instance, we note that the k_T-jet matching scheme that we have used is rather like the phase space slicing method in NLO calculations and we wonder if it might be replaced by something that is more like the subtraction method. Additionally, it would be good to incorporate knowledge of one loop splitting functions into the algorithm so as to better match the one loop perturbative calculations.

We have not constructed computer code that could demonstrate that the algorithm presented here works in practice. There may well be undesirable features that show up when we have real code to test. However, the algorithms presented here are a refined version of the algorithm presented in refs. [8, 9]. This cruder algorithm does exist as code and, when coupled appropriately to PYTHIA, appears to work quite well [10].

Acknowledgments

We are grateful to S. Mrenna for advice, particularly concerning the advisability of dealing with the hardest splitting first. This work was supported in part the United States Department of Energy and by the Swiss National Science Foundation (SNF) through grant no. 200021-101874 and by the Hungarian Scientific Research Fund grants OTKA T-038240.

References

[1] G. Marchesini et al., *HERWIG: a Monte Carlo event generator for simulating hadron emission reactions with interfering gluons. Version 5.1 - April 1991*, *Comput. Phys. Commun.* **67** (1992) 465;

G. Corcella et al., *Herwig 6: an event generator for hadron emission reactions with interfering gluons (including supersymmetric processes)*, *JHEP* **01** (2001) 010 [hep-ph/0011363].

[2] T. Sjostrand, *The Lund Monte Carlo for jet fragmentation and e^+e^- physics: jetset version 6.2*, *Comput. Phys. Commun.* **39** (1986) 347;
T. Sjostrand et al., *High-energy-physics event generation with PYTHIA 6.1*, *Comput. Phys. Commun.* **135** (2001) 238 [hep-ph/0010017].

[3] Z. Nagy and Z. Trócsányi, *Group independent color decomposition of next-to-leading order matrix elements for $e^+e^- \to$ four partons*, *Phys. Lett.* **B 414** (1997) 187 [hep-ph/9708342];
Z. Nagy and Z. Trócsányi, *Next-to-leading order calculation of four-jet observables in electron positron annihilation*, *Phys. Rev.* **D 59** (1999) 014020 [hep-ph/9806317], erratum *ibid.* **D62** (2000) 099902;
Z. Nagy and Z. Trócsányi, *Multi-jet cross sections in deep inelastic scattering at next-to-leading order*, *Phys. Rev. Lett.* **87** (2001) 082001 [hep-ph/0104315];
Z. Nagy, *Three-jet cross sections in hadron hadron collisions at next-to-leading order*, *Phys. Rev. Lett.* **88** (2002) 122003 [hep-ph/0110315]; *Next-to-leading order calculation of three-jet observables in hadron hadron collision*, *Phys. Rev.* **D 68** (2003) 094002 [hep-ph/0307268].

[4] J.M. Campbell and R.K. Ellis, *An update on vector boson pair production at hadron colliders*, *Phys. Rev.* **D 60** (1999) 113006 [hep-ph/9905386]; *Radiative corrections to z b anti-b production*, *Phys. Rev.* **D 62** (2000) 114012 [hep-ph/0006304];
Next-to-leading order corrections to W + 2jet and Z + 2jet production at hadron colliders, *Phys. Rev.* **D 65** (2002) 113007 [hep-ph/0202176].

[5] S. Frixione and B.R. Webber, *Matching NLO QCD computations and parton shower simulations*, *JHEP* **06** (2002) 029 [hep-ph/0204244].

[6] S. Frixione, P. Nason and B.R. Webber, *Matching NLO QCD and parton showers in heavy flavour production*, *JHEP* **08** (2003) 007 [hep-ph/0305252].

[7] S. Frixione and B.R. Webber, *The MC@NLO 2.3 event generator*, hep-ph/0402116.

[8] M. Kramer and D.E. Soper, *Next-to-leading order QCD calculations with parton showers. i: collinear singularities*, *Phys. Rev.* **D 69** (2004) 054019 [hep-ph/0306222].

[9] D.E. Soper, *Next-to-leading order QCD calculations with parton showers, II. Soft singularities*, *Phys. Rev.* **D 69** (2004) 054020 [hep-ph/0306268].

[10] M. Krämer, S. Mrenna and D.E. Soper *Next-to-leading order QCD jet production with parton showers and hadronization*, hep-ph/0509127.

[11] S. Catani, F. Krauss, R. Kuhn and B.R. Webber, *QCD matrix elements + parton showers*, *JHEP* **11** (2001) 063 [hep-ph/0109231];
S. Mrenna and P. Richardson, *Matching matrix elements and parton showers with herwig and pythia*, *JHEP* **05** (2004) 040 [hep-ph/0312274].

[12] S. Catani and M.H. Seymour, *A general algorithm for calculating jet cross sections in NLO QCD*, *Nucl. Phys.* **B 485** (1997) 291 [hep-ph/9605323], erratum *ibid.* **B510** (1997) 503.

[13] Y.L. Dokshitzer, in *Workshop on jet studies at LEP and HERA*, Durham, 1990, reported in W.J. Stirling, *Hard QCD working group: theory summary J. Phys.* **G 17** (1991) 1567;
S. Catani, Y.L. Dokshitzer, M. Olsson, G. Turnock and B.R. Webber, *New clustering algorithm for multi - jet cross-sections in e^+e^- annihilation*, *Phys. Lett.* **B 269** (1991) 432;

see also S. Bethke, Z. Kunszt, D.E. Soper and W.J. Stirling, *New jet cluster algorithms: next-to-leading order QCD and hadronization corrections*, *Nucl. Phys.* **B 370** (1992) 310, erratum *ibid.* **B523** (1998) 681.

[14] T. Sjostrand and P.Z. Skands, *Transverse-momentum-ordered showers and interleaved multiple interactions*, *Eur. Phys. J.* **C 39** (2005) 129 [hep-ph/0408302].

[15] P. Nason, *A new method for combining NLO QCD with shower Monte Carlo algorithms*, *JHEP* **11** (2004) 040 [hep-ph/0409146].

[16] See R.K. Ellis, W.J. Stirling and B.R. Webber, *QCD and collider physics*, *Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol.* **8** (1996) 1.

[17] L. Lonnblad, *Correcting the colour-dipole cascade model with fixed order matrix elements*, *JHEP* **05** (2002) 046 [hep-ph/0112284].

[18] S. Mrenna and P. Richardson, *Matching matrix elements and parton showers with HERWIG and PYTHIA*, *JHEP* **05** (2004) 040 [hep-ph/0312274].

JHEP10(2005)024